# Indefinite Integrals from Integration

Walter G. Spunde

**Department of Mathematics and Computing**
**University of Southern Queensland**

# The proposition:

**Escape from the function as formula syndrome**

**Provide realistic experience with functions**

Using computing power on numeric lists of related variables, students can tabulate approximate values for (line) integrals.

Approximate derivatives and integrals and arithmetic combinations of such functions, whose values are known only from tabulations, can be computed and manipulated.

Besides providing useful practical experience with *computable* functions whose values cannot be obtained from the elementary functions, this approach highlights the definition of the integral as a *function*.

# Function values as outputs from algorithms


$$\int_a^b g(x)\ dx \quad ; \ y' = g(x, y)$$

Where do you get a *function* out of that?

When algorithms for function values are always one line algebraic formulae; when functions are always *given*, and always given as one line algebraic formulae; when working with functions means working with one line algebraic formulae; one can only conclude that functions ***are*** one line algebraic formulae (and function values are *functions at* some point.)

# School yard differentiation and integration

Differentiation is getting one formula from another following a given set of rules. Since a function is a formula, you're getting one function from another. Integration is the opposite.

$$(d/dx)\, f \rightarrow f' \qquad S\, f' \rightarrow f$$

The derivative *at a point* gives you the slope —– sort of like $f(x)$ when $x = 1$ —– but $f(1)$ ?

[$f(x)$ is like the *name* of the function.]

Integration is getting an area under a curve -- *i.e.* a number.

[Does any one talk about the integral to a point?]

If integral means anti-derivative what's so marvellous about finding that the derivative of the anti-derivative is the function (formula) back again?

# Instructors avoiding the indefinite integral -- in the Fundamental Theorem

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

Which form is usually presented to students ?

$$f(x) = \int_a^x f'(t)\,dt + C$$

How hard is it to get that first form back later ?

# Indefinite integral ≠ anti-derivative

$$f_1(x) = \int_a^x g(t)\, dt$$

$$f_2(x) = \int_c^x g(t)\, dt$$

$$f_2(x) = f_1(x) + C$$

We distinguish the *indefinite integral* from the *anti-derivative*.
The usage here is that an integral is *indefinite* if it has an
indefinite starting point.

An integral is a function giving the accumulated value of a variable distributed with a density given by the integrand.

It is the cumulative **sum of the products** of *average density values* over intervals by the *interval widths*.

$$f(x) = \sum_{k=1}^{n} f(\xi_k)\Delta t_k$$

$$\text{where} \quad \sum_{k=1}^{n} \Delta t_k = x - a$$

# Avoiding the indefinite integral in integration by parts

| |
|---|
| $$\int_a^x g(t)\,f'(t)\,dt = \big[g(t)\,f(t)\big]_a^x - \int_a^x g'(t)\,f(t)\,dt$$ |
| $$\int_a^b g(t)\,f'(t)\,dt = \big[g(t)\,f(t)\big]_a^b - \int_a^b g'(t)\,f(t)\,dt$$ |
| $$\int g(x)\,f'(x)\,dx = g(x)\,f(x) - \int g'(x)\,f(x)\,dx$$ |

Which form would you present to your students ?

# Avoiding the indefinite integral in separable differential equations

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int_a^X \frac{1}{g(y)} \frac{dy}{dx} dx = \int_a^X f(x)\, dx$$

$$\int \frac{1}{g(y)} dy = \int f(x)\, dx$$

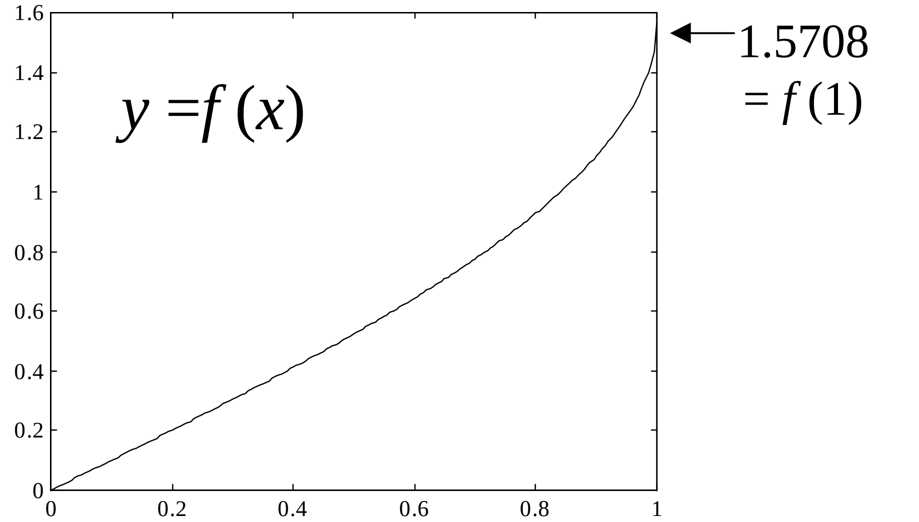
Which form would you present to your students ?

Can they cope with both forms?

# Avoiding the indefinite integral in *improper* integrals

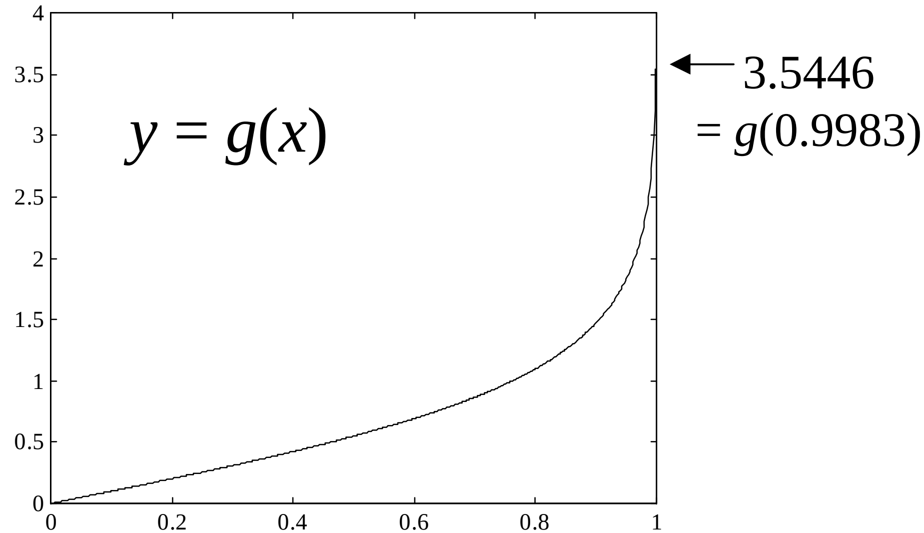
$$\int_0^1 \frac{1}{\sqrt{1-t^2}}dt = f(1)$$

$$\text{where} \quad f(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}dt$$




$$\int_0^1 \frac{1}{1-t^2}dt = g(1) \qquad (?)$$

$$\text{where} \quad g(x) = \int_0^x \frac{1}{1-t^2}dt$$




To make $g(x) = 10$ we need $x$ to be 0.99999999587769

# Being unaware of cumulative summation

The first reaction to tabulating an integral, using Simpson's Rule, in order to plot a graph of an integral, is often that a separate Simpson's rule would have to be applied to compute the integral's value at every point.

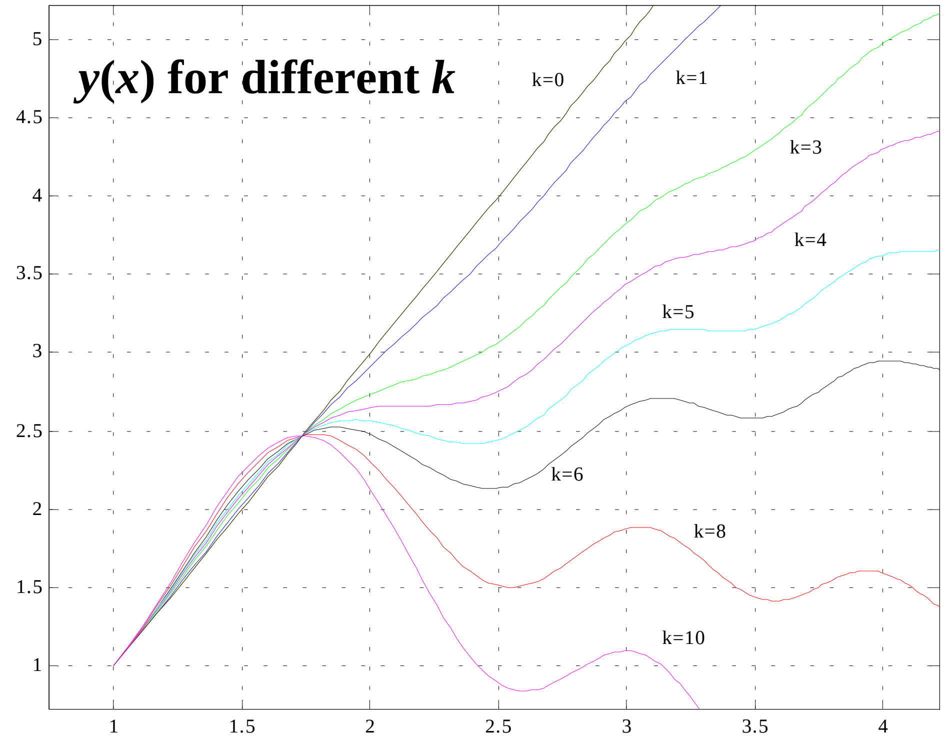
Numerical techniques are not presented as *accumulation*.

Given tabulation points $x_0, x_1, x_2, \ldots$
with function values $y_0, y_1, y_2, \ldots$ integral values for parabolic approximations are obtained from: $I(x_0) = 0$ and for $k = 0,1,2\ldots$

$$I(x_{2k+1}) = \mathrm{I}(x_{2k}) + h\,(5\,y_{2k} + 8\,y_{2k+1} - y_{2k+2})/12$$

$$I(x_{2k+2}) = \mathrm{I}(x_{2k+1}) + h(-y_{2k} + 8\,y_{2k+1} + 5y_{2k+2})/12$$

# Putting the indefinite integral off (indefinitely?)




$$f(x) = \int_a^x e^{-t^2}\, dt$$

$$y'' = k\cos(x^2);\quad y(1) = 1, y'(1) = 2;$$

The integral has a zero at 1.73625

$$\Rightarrow y(x) = -1 + 2x + k\int_1^x dt \int_1^t \cos s^2 ds$$

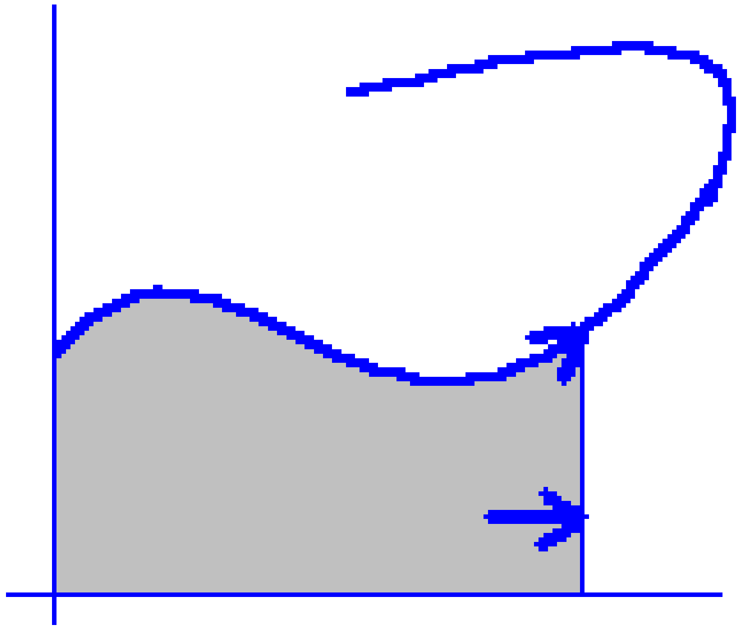
Can students *differentiate* and *integrate* these functions? *Graph* them? *Combine* them with other functions?

# Re-inforcing possible misconceptions

The integral is a cumulative sum –– not *necessarily* an **area**.

Distance travelled along a curve or the time taken to get to any point on a curve are more intuitive ideas than cumulative area under a curve (of the graph of a functional relationship).

Avoiding arclength is an **algebraic imperative**.



# Leibniz's *d* and *S*    Our *sample* and *mids*

$d\,(2, 5, 9, 11) = 3, 4, 2$

$S\,(2, 5, 9, 11) = 0, 2, 7, 16, 27$

$d\,S\,(2, 5, 9, 11) = 2, 5, 9, 11$

$S\,d\,(2, 5, 9, 11) = 0, 3, 7, 9$

$= 2, 5, 9, 11 - 2$

$x = sample(a, b, n)\,;\; y = f\,(x)$

$dx = d(x)\;;\; dy = d(y)$

$dy/dx$ is a sample of $y'$ values
at some intermediate points, $\xi$

$S\,f\,(\xi)\,dx$ is the integral
(using some intermediate points).

$sample(0, 1, 4)$

$= 0, 0.25, 0.5, 0.75, 1$

$mids(0, 0.5, 1) = 0.25, 0.75$

Put $\xi = mids\ x$ or $m\ x$
for some averaging function, $m$.

Plot $dy/dx$ vs $\xi$

Compute $I = S\ f\,(\xi)\,dx$

or $I = S\,(\,mids\ y)\,dx$

or $I = S\,(\,f\ m\ x)\,dx$

or $I = S\,(\,m\ f\ x)\,dx$

Plot $I$ vs $x$

# Parametric curves and areas of regions

$t = sample(a, b, n)$ ;

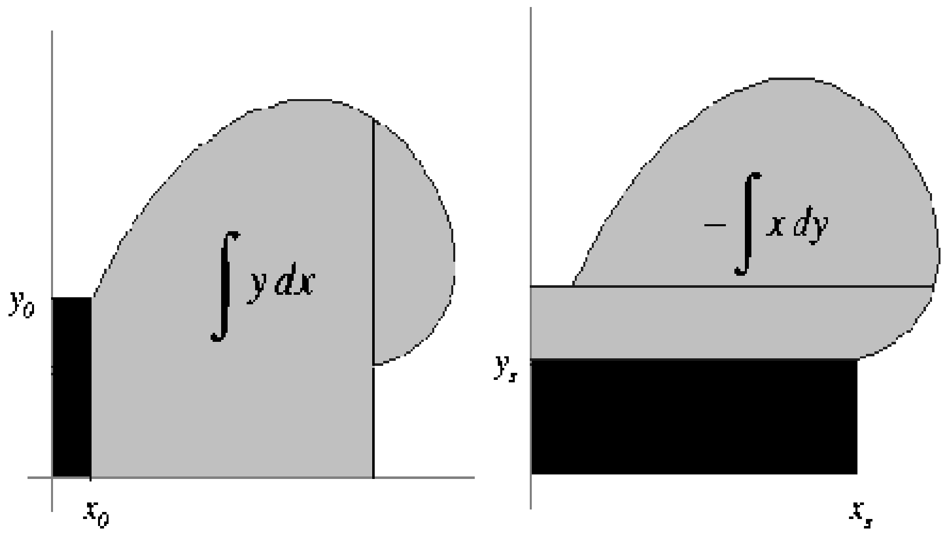
$x = f(t)$ ; $y = g(t)$

$dx = d(x)$ ; $dy = d(y)$

$s = S\ sqrt\ (dx^2 + dy^2)$

$A = S\ (\ mids\ \ y)\ dx$

$x_0 y_0 + S\ y\ dx = x_s y_s - S\ x\ dy$




Integration by Parts

**Conclusion: Two ways of coming to an understanding of relationships defined by differential equations might help develop the concept of a *function* better.**

$$y' = f(x)$$

Algebraic manipulation
to arrive at a formula.

Numerical computation
to arrive at a tabulation

$$y = S\, f(\xi)\, dx$$