THE PARABOLIC ARC:
A SUBTLE STEP FORWARD

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SUMMARY

The paper shows the elementary use of a computer algebra system (CAS) to avoid the algebraic complexities involved in extending piece-wise linear approximations to parabolic ones and points the way forward to spline or to Taylor series approximations. The discussion encourages students to view function values as outputs from possibly complex algorithms rather than simple algebraic formulae.

INTRODUCTION

Numerical solutions to problems are the norm in a world where powerful computational tools are readily accessible, while closed form expressions, in terms of a short list of elementary and special functions, are more likely to be used to develop numerical approximation techniques for a solution than provide formulae for the solution itself of any but the simplest problems. An attempt to reflect this sort of approach at an elementary level might focus on straight lines, parabolas and other polynomials as relevant to describing local approximations to functions rather than as entities whose properties on a global scale are the chief concern.

The popular computer pastime in a mathematics class of zooming in on graphs serves an instructive purpose. It shows the local linearity that justifies not only the drawing of the graph itself by joining points, but also difference quotient approximations to derivatives, Newton's method for approximating zeros, the trapezoidal approximation to the integral, Euler's method for approximating the solutions of a differential equation and L'Hospital's rule for comparing the relative sizes of functions [1,2,3]. Lines that look straight, however, may, in fact, be curved (see Figure 4) and the difference could be the difference between a rough approximation and a superb one.

LOCAL PARABOLIC APPROXIMATION

A piece-wise linear approximation to the values of a function (of a single variable) whose values are known, or computable, at a finite set of variable values consists of a set of straight lines joining successive pairs of grid points. A piece-wise parabolic approximation joins successive triplets of points with parabolas. Since the algebraic manipulation will be done by a CAS, the simplification of a uniform grid is unnecessary and we will refer to any triplet of adjacent tabulation points as $x\text{-}h$, x and $x+H$.

With $f(x\text{-}h)=u$; $f(x)=v$; $f(x+H)=w$ the coefficients of the parabola $y=c+bx+ax^2$ that passes through the points $(x\text{-}h,u)$, (x,v) and $(x+H,w)$ can be obtained by solving the three linear equations

$$c + b(x\text{-}h) + a(x\text{-}h)^2 = u$$

$$c + bx + ax^2 = v$$

$$c + b(x+H) + a(x+H)^2 = w$$

for the unknowns, a,b,c .

A computer algebra system delivers the result:

$$c = \frac{1}{h+H}\left\{x(x-h)\frac{w-v}{H} - x(x+H)\frac{v-u}{h} + v(h+H)\right\}$$

$$b = \frac{1}{h+H}\left\{(h-2x)\frac{w-v}{H} + (H+2x)\frac{v-u}{h}\right\}$$

$$a = \frac{1}{h+H}\left\{\frac{w-v}{H} - \frac{v-u}{h}\right\}$$

A plot of the parabolic arcs that constitute the approximation will reinforce the idea that split definition functions are by no means to be regarded as esoteric oddities but indeed useful and common in practice.

DERIVATIVES (Divided Differences)

The derivative of the parabolic approximating function at x follows easily by substituting the solutions for b and a into $b+2ax$ to get

$$\frac{h\dfrac{w-v}{H} + H\dfrac{v-u}{h}}{h+H}$$

the weighted average of the right and left difference quotients. The central difference approximation, $\frac{w-u}{2h}$, follows easily by setting $H = h$. It may be worth observing that a symbolic manipulator will probably not deliver the above arrangement of terms, but rather something like $\frac{h^2w - h^2v - H^2u + H^2v}{hH(h+H)}$ so the task of re-casting results from algebraic manipulations into interpretable patterns is unlikely to be automatic and will remain a challenge for students.

INTEGRALS (A Cumulative Simpson's Rule)

The area under the parabola $y = c + bx + ax^2$ between the tabulation points $x-h$ and x can be found by anti-differentiation and a computer algebra system will return:

$$\frac{h\left\{h^2(v-w) + 3H^2(u+v) + 2hH(u+2v)\right\}}{6H(h+H)}$$

while the area between x and $x+H$ is given by

$$\frac{H\left\{H^2(v-u) + 3h^2(w+v) + 2hH(w+2v)\right\}}{6h(h+H)}$$

For uniform subdivisions these area computations reduce to simple formulae that can be displayed as the matrix product

$$\frac{h}{12}\begin{bmatrix} 5 & 8 & -1 \\ -1 & 8 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Clearly the sum of these areas is the dot product $\frac{h}{3}(1,4,1)\bullet(u,v,w)$ which adds up to the familiar Simpson's Rule over the entire interval.

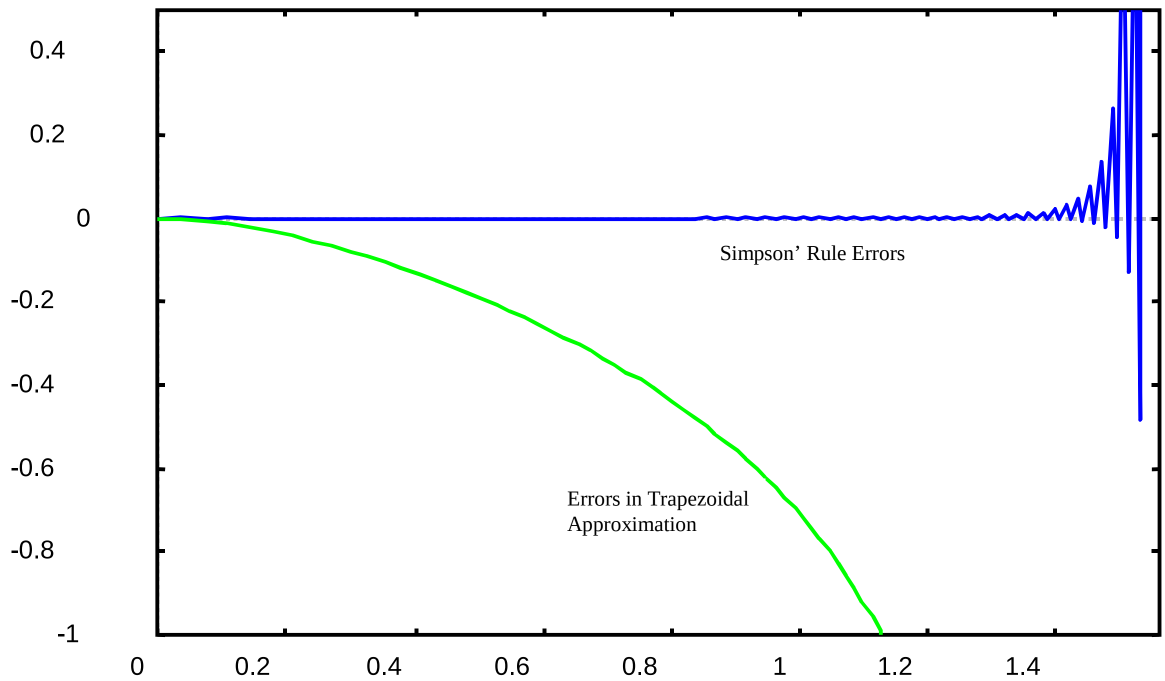

Figure 1: Errors in cumulative Simpson's and Trapezoidal integrations

In order to tabulate and graph an integral of the function being approximated, the partial sums to each tabulation point should be accumulated. For an odd number of subdivisions, we need to compute the area under a parabola fitted to the last three points of the data set but between only the last two points, ie. from x to $x+H$. The option of adding this value to the tabulation, when required, is easily programmed into the procedure. Thus, given an adequate sample of data points, one will be able to graph an integral of a function known at a given set of tabulation points (and perhaps known only at these points). The order of magnitude improvement in accuracy is apparent from Figure 1.

EULER'S METHOD

The extension of Euler's Method for approximating solutions of the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0$$

is somewhat different to the previous examples, as here there is information about the function at only one point. The approximating polynomial is obtained, by fitting the function's value, slope and curvature to the parabola:

$$a + bx_0 + cx_0^{\ 2} = y_0\ ; \quad b + 2cx_0 = f(x_0, y_0)\ ; \quad 2c = f_x(x_0, y_0) + f_y(x_0, y_0)\, f(x_0, y_0)$$

(Knowledge of partial differentiation is only required for a general presentation: second derivatives for specific examples can be found with implicit differentiation.) It is easy to see, even without a symbolic manipulator, that at $x = x_0$ the parabola will be at $y_0 + f(x_0, y_0)h + ch^2$. Thus we obtain the iterative procedure, (that need not involve equal step sizes):

$$(x_{n+1}, \quad y_{n+1}) = (x_n + h_n, \quad y_n + f(x_n, y_n)h_n + ch_n^{\ 2})$$

where, theoretically at least,

$$c_n = \frac{1}{2}\left\{\frac{\partial f}{\partial x}(x_n, y_n) + \frac{\partial f}{\partial y}(x_n, y_n) f(x_n, y_n)\right\}$$

The refinement, like that of Simpson's rule over the trapezoidal, is quite spectacular with accuracies at a different order of magnitude.

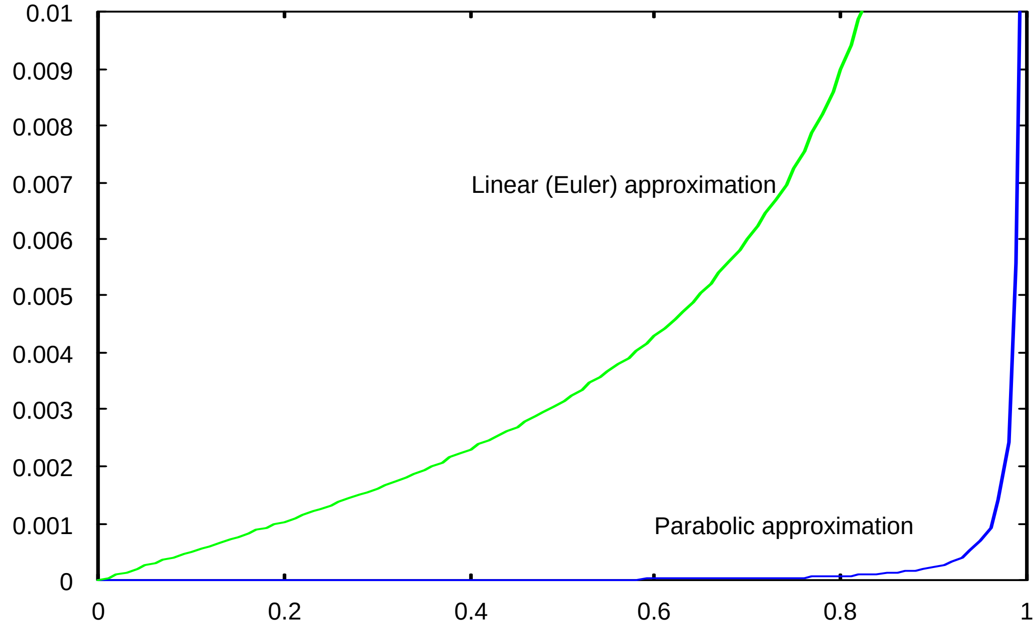

Figure 2 : Errors in iterative solution of $y' = -x / y, \quad y(0) = 1$
by Euler's linear method and quadratic Taylor approximation.

ENDPOINT SLOPE PROBLEMS

The success of the parabolic extension to Euler's method might suggest that a parabolic estimate of the slopes at the end-points of an interval might be preferable to the left and right hand difference quotients usually used at the interval ends in the central difference approximation. Here, however, we are in for a something of a disappointment. Proceeding as before to fit a parabola to pass through the points $(x\text{-}h,u)$, (x,v) and $(x+H,w)$, the slopes at the end-points are given by:

$$\frac{1}{h+H}\left\{(2h+H)\frac{v-u}{h} - h\frac{w-v}{H}\right\}$$

where u,v,w are the first three function values, and

$$\frac{1}{h+H}\left\{(2H+h)\frac{w-v}{H} - H\frac{v-u}{h}\right\}$$

where u,v,w are the last three function values.

In the case of a uniform partition, these formulae simplify to

$\frac{3(v-u)-(w-v)}{2h}$ and $\frac{3(w-v)-(v-u)}{2h}$

For the sine function it appears that the parabolic approximation is, in fact, worse than the linear approximation. With 100 sub-divisions of the interval (0,), the slope at $x = 0$ is given by the linear approximation as 0.99983551 which is in error by 0.000164. The parabolic approximation at $x = 0$, $1.0003289x$-$0.015704088x^2$, however, has an error of modulus 0.0003289, twice as large as in the linear case.

The approximation, whether linear or parabolic, seems to be fundamentally flawed at the ends, since repeated approximate differentiation leads to catastrophic failure at the ends without similar behaviour in the interior of the interval.

This exercise highlights the difference between two types of polynomial approximation. It appears that while a parabola fitted to three points of a function may approximate that function much more closely than a straight line, as shown by Simpson's rule, its slope may nevertheless be locally quite different from the function it approximates. The Euler approach ensures equality of slopes and curvatures at a particular point. This option is not available for the end-point derivative problem, but, for Newton's Method there is a choice.

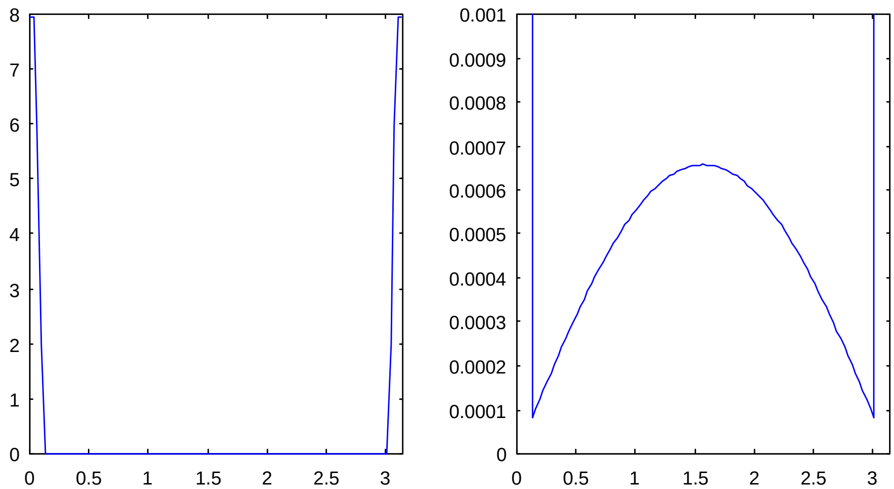

Figure 3: Difference between $\sin x$ and $\frac{d^4 \sin x}{dx^4}$ as computed with central difference approximations linear at the ends. (End point slopes from parabolas are worse.)

NEWTON'S METHOD

Either of the two basically different approaches to parabolic approximation used in the previous exercises could be applied to extend Newton's Method. The apparent complications that arise from the fact that a parabola may not even cut the axis are usually enough to discourage further investigation, but on reflection, they simply mirror considerations that should be addressed in any application of Newton's Method as well. Since Newton's Method nearly always delivers a sequence of approximations, however, it is common to generate this sequence and then marvel at any unusual behaviour. Whether linear or parabolic approximations are used, the basic requirement is that the approximation faithfully reflect the behaviour of the function in the region of the zero being sought.

One way to attempt to ensure this is to find an interval over which the function changes sign (once). A parabola that passes through the function values at the end points of such an interval and a third point, say the midpoint of the interval, should produce a reasonably good approximation to the function.

If we set x to be the midpoint of the interval, with the endpoints at $x \pm h$ the coefficients of a parabola passing through $(x\text{-}h,u)$, (x,v) and $(x+H,w)$ can be found using a symbolic manipulator as before. The extreme of this parabola occurs at $m = x + h(w-u)/D$ where $D = (v-u)-(w-v)$ and u and w are of opposite sign.
The distance of both zeros of the parabola from the extreme point is (the modulus of) $d = h\sqrt{8vD + (w-u)^2}\ /(2D)$ and whether this quantity should be added or subtracted from m depends on the sign of $\{(w-u)^2 - 8vD\}(w-u)/D$. Successive steps in the iteration require the same choice of new interval as required for the bisection method.

The results however are disappointing. If one end of the interval is very close to the zero, it may take a number of iterations to improve the approximation significantly since the curvature of the parabola is (unhelpfully) affected by the two points not so close to the zero.
The alternative approach is to use a parabolic approximation that agrees with the function in its value, slope and curvature at the approximate zero, X . Thus we will need to solve for parabola coefficients at every iteration point and the resulting quadratic equation for the zero closest to X . The same greatly improved accuracy with each iteration is apparent as in the Euler method (Figure 4). Often only one or two iterations are required for 6 digit accuracy, while the linear Newton's method might require four or five iterations. It might also be noted that since the derivatives of f are required only at X , these may be obtained by differentiation arithmetic rather than symbolic differentiation and substitution [4].

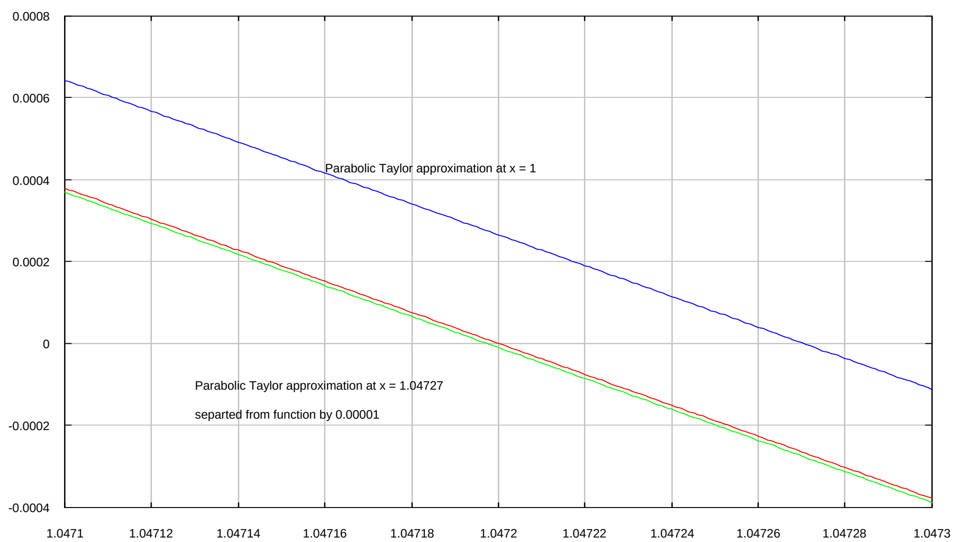

Figure 4 : Quadratic Taylor approximation in Newton's Method for a zero of $e^{-x/4}(0.5x^3 + x^2 - 2.4x + 2)\sin 3x$

CONCLUSIONS

The value in developing the parabolic approximations described lies not only in demonstrating the order of magnitude improvements in accuracy that result from higher degree polynomial approximations, but also the superiority of a Taylor approach to polynomial approximation over the point fitting approach. The concrete numeric experiences that students are exposed to in working through these approximations probably outweigh the value of the techniques as tools for the tasks at hand. Later work can show how the characteristic behaviour of high degree polynomials undermines the usefulness of fitting polynomials to points while, on the other hand, strikingly reveals where Taylor series approximations begin to fail.

The exercises also demonstrate the elementary use of computer algebra systems for the development of algorithms without tedious and time-wasting algebraic manipulation and highlight the fact that finding patterns in results with a geometric or other interpretation will not be done by the machine. Nevertheless, with word processing systems that incorporate the results of mathematical operations, the phrase *clearly, it can be seen that* may well acquire a whole new world of meaning.

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