Evaluation of Tweedie family densities using inversion and acceleration techniques

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Introduction

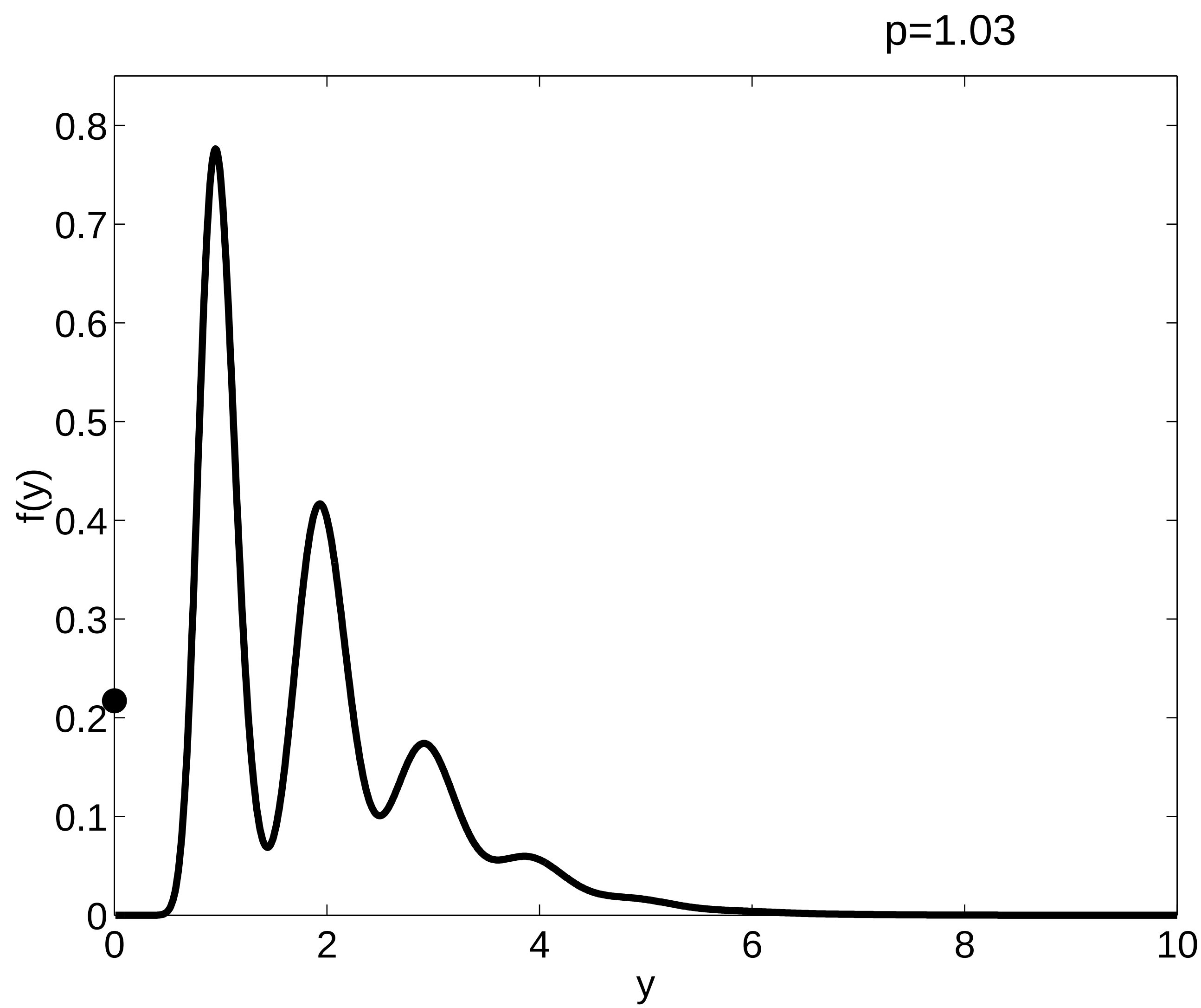
1. Properties and uses of Tweedie densities.
2. Methods of evaluation for Tweedie densities.
3. One method: Inverting the cgf.
4. Fast evaluation for applications.

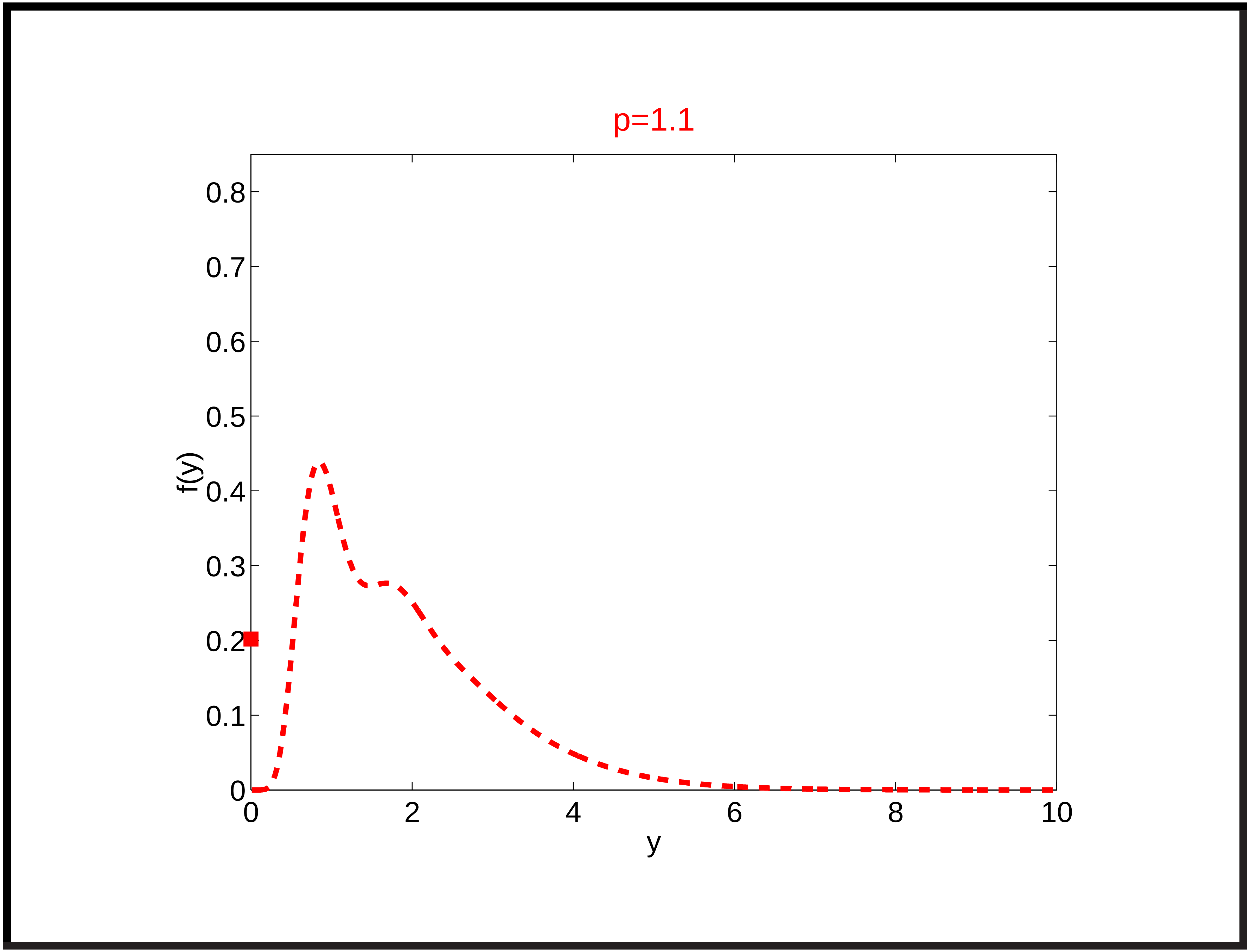
Properties of Tweedie densities

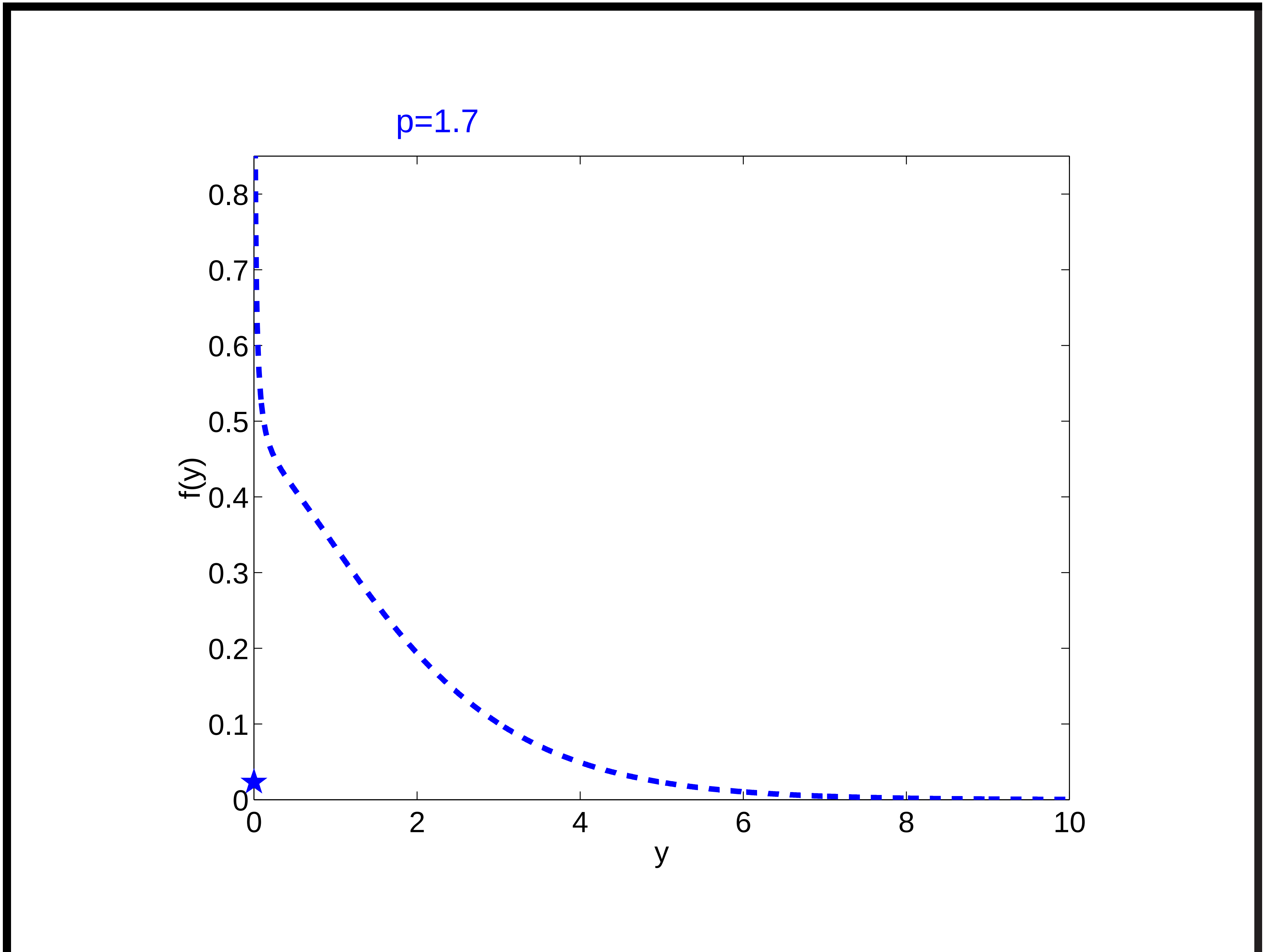
- Characterized by *index* $p \notin (0,1)$.
- Mean: $E[Y] = \mu$.
- var$[Y] = \phi\mu^p$ ($\phi > 0$ is *dispersion parameter*).
- – $p = 0$: normal ($\phi = \sigma^2$);
  – $p = 1$: Poisson ($\phi = 1$);
  – $p = 2$: gamma;
  – $p = 3$: inverse Gaussian.

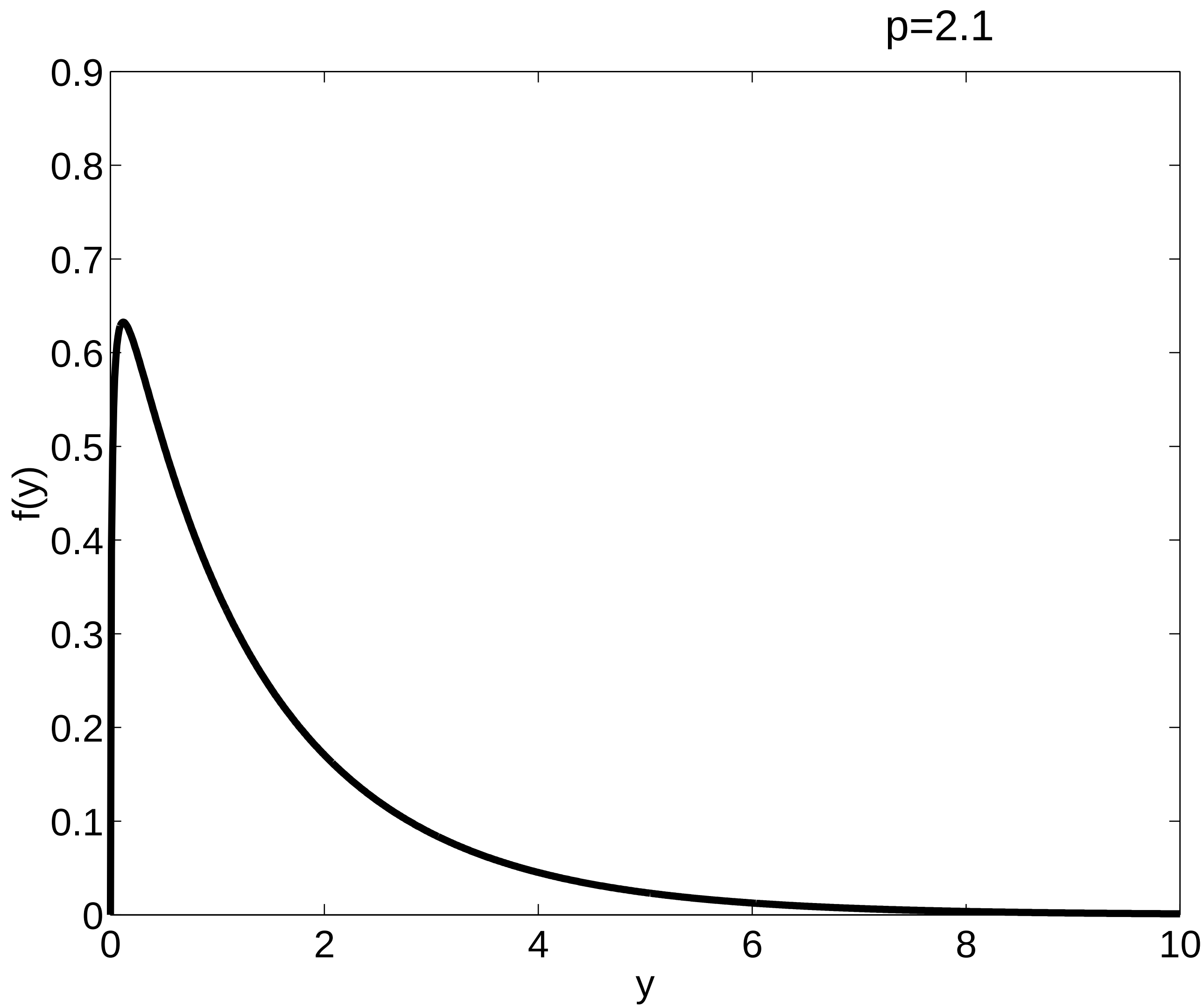
Range of p	Distribution Type
$p \leq 0$	*continuous* on $(-\infty, \infty)$
$0 < p < 1$	(No densities exist)
$p = 1$	non-negative *discrete*
$1 < p < 2$	*mixed* on $y \geq 0$ (mass at $y = 0$)
$p \geq 2$	*continuous* on $(0, \infty)$

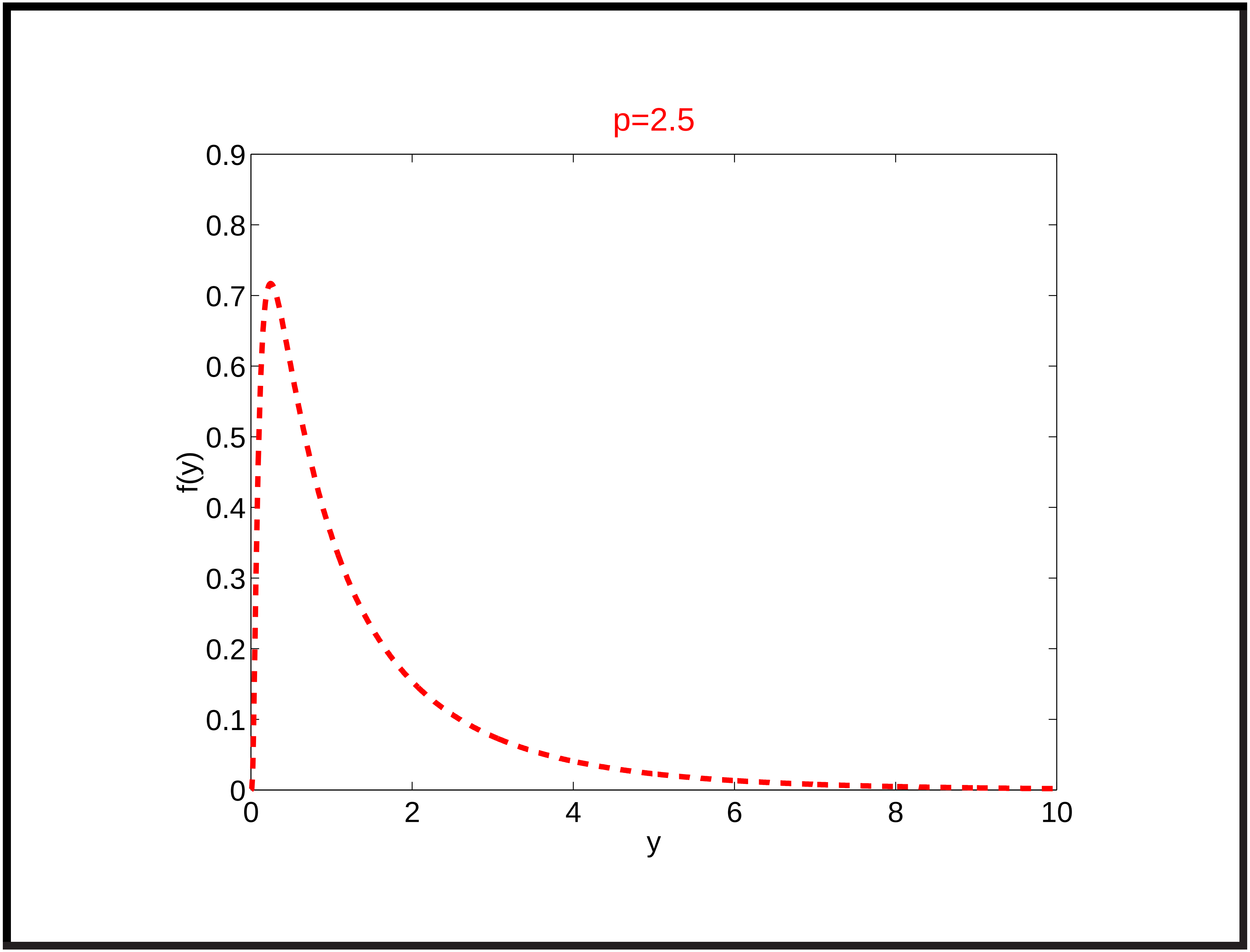
- Hence many practical uses.
- Rarely used since evaluation is difficult.

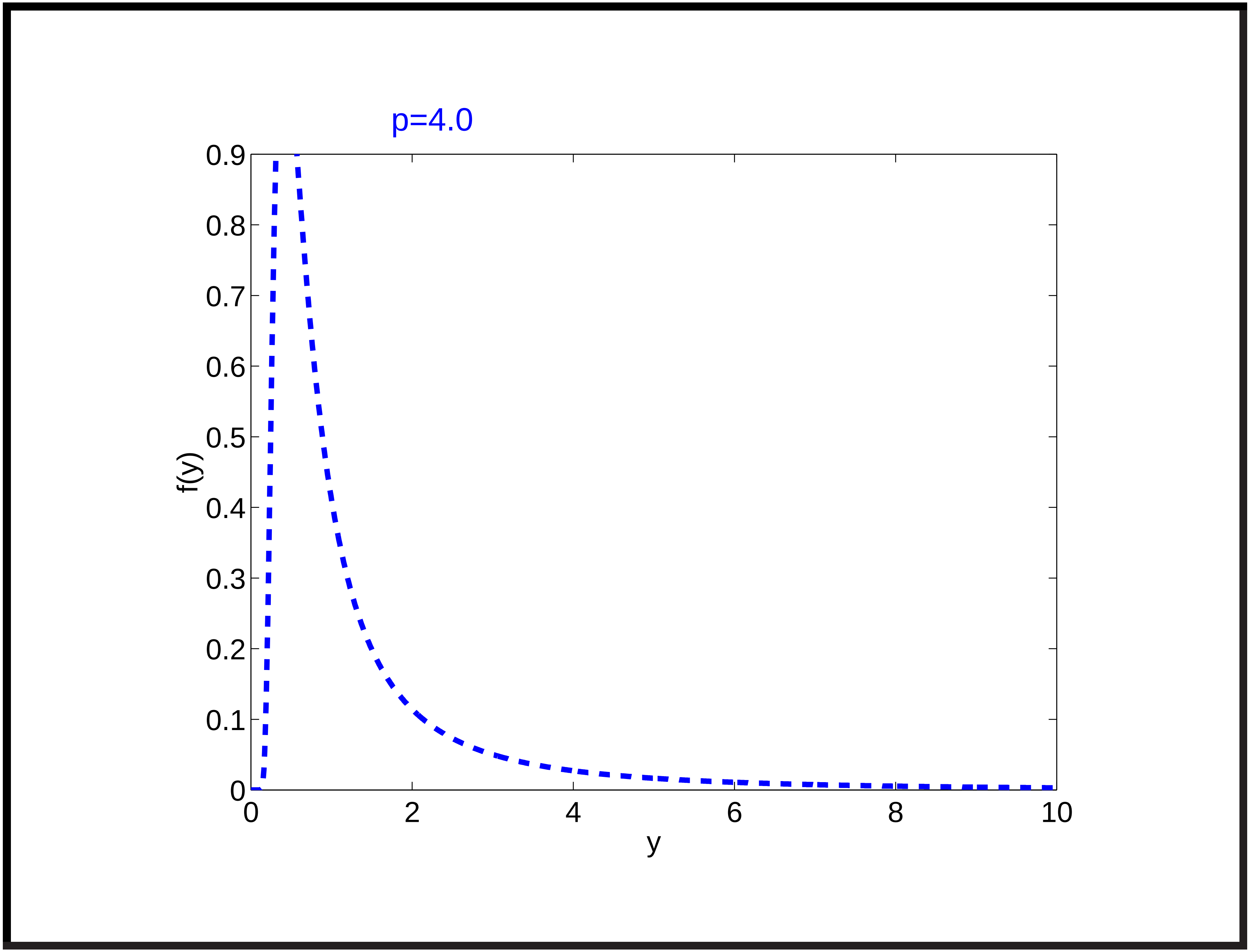

p=1.03
f(y)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
y


p=1.1
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
f(y)
0
2
4
6
8
10
y


p=1.7
f(y)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
y


p=2.1
f(y)
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0
2
4
6
8
10


p=2.5
f(y)
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
2
4
6
8
10


p=4.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
f(y)
0
2
4
6
8
10
y

Definition of Tweedie Densities

- Densities are *exponential dispersion models*:

$$f_p(y;\mu,\phi) = a_p(y,\phi)\exp\left\{\frac{1}{\phi}[y\theta - \kappa(\theta)]\right\},$$

where $\theta = (\mu^{1-p}-1)/(1-p)$;
$\kappa(\theta) = (\mu^{2-p}-1)/(2-p)$.

- $a_p(y,\phi)$ generally has *no closed form*.

- Tweedie densities are also *dispersion models*:

$$f_p(y;\mu,\phi) = b_p(y,\phi)\exp\left\{-\frac{1}{2\phi}d(y,\mu)\right\}$$

where $d(y,\mu)$ is the (unit) *deviance*.

- $b_p(y,\phi)$ generally has *no closed form*.
- Simple *moment generating functions* (mgf) and *cumulant generating functions* (cgf).

Approaches to Evaluation

1. Series expansions:

   An infinite summation.

2. Inverting cumulant generating function (cgf):

   An infinite oscillating integral.

3. Saddlepoint approximation:

   Only in limiting cases.

Inversion of the cgf

- $$f_p(y;\mu,\phi) = a_p(y,\phi)\exp\left\{\frac{1}{\phi}[y\theta - \kappa(\theta)]\right\},$$

  where $\theta = (\mu^{1-p} - 1)/(1-p)$;
  $\kappa(\theta) = (\mu^{2-p} - 1)/(2-p)$.

- With $\mu = 1$, we have $f_p(y;\mu = 1,\phi) = a_p(y,\phi)$.

- So only need evaluate $a_p(y,\phi)$ (that is, at $\mu = 1$).

- Inversion formula (continuous case):

$$a_p(y,\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[\kappa(it) - ity]\,dt.$$

with $\kappa(t)$ the cgf (when $\mu = 1$), and $i = \sqrt{-1}$.

- Work with *real* parts:

$$a_p(y,\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[\Re k(t)]\cos \Im k(t)\,dt,$$

where $k(t) = \kappa(it) - ity$.

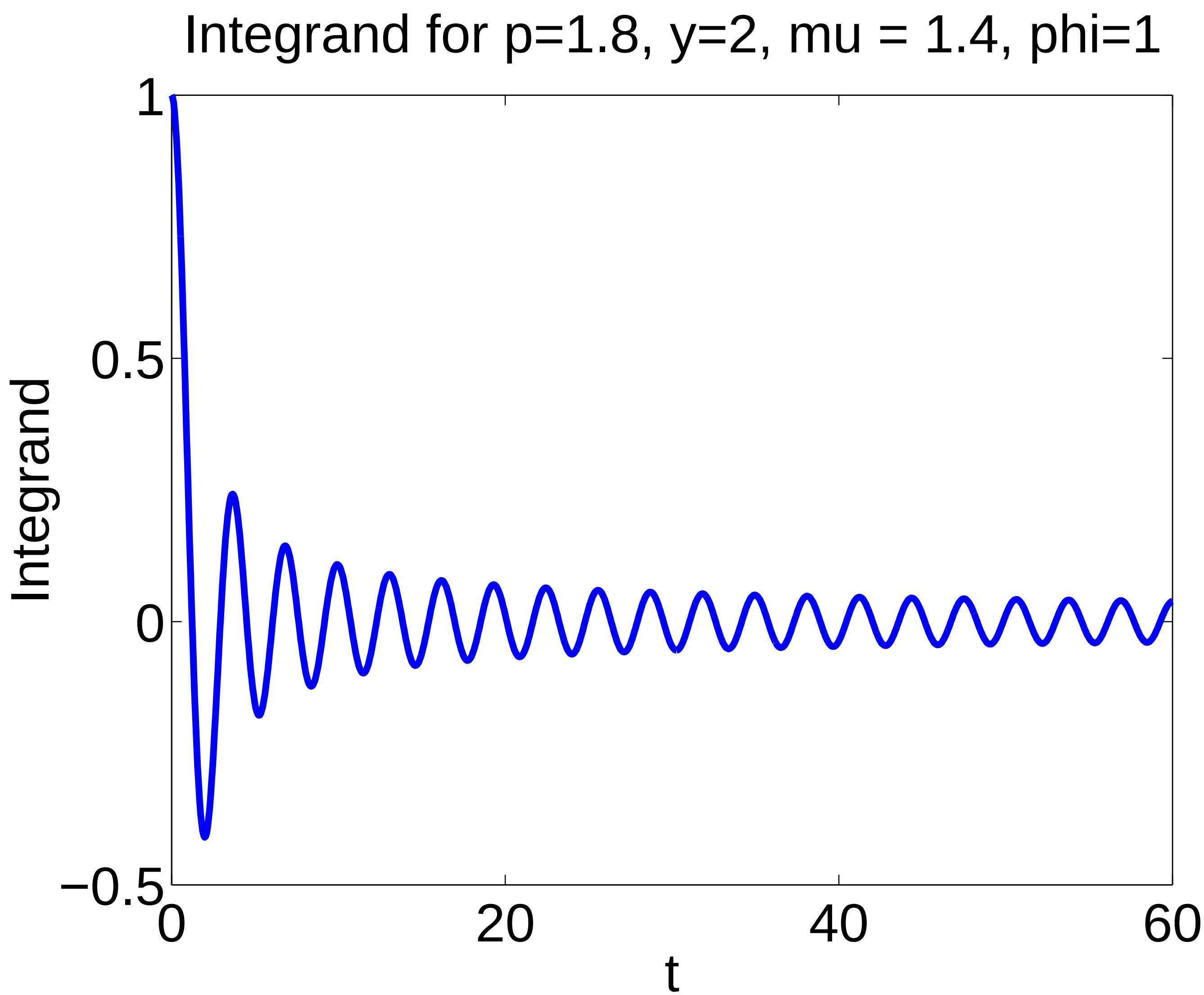
Inversion Formula for $1 < p < 2$

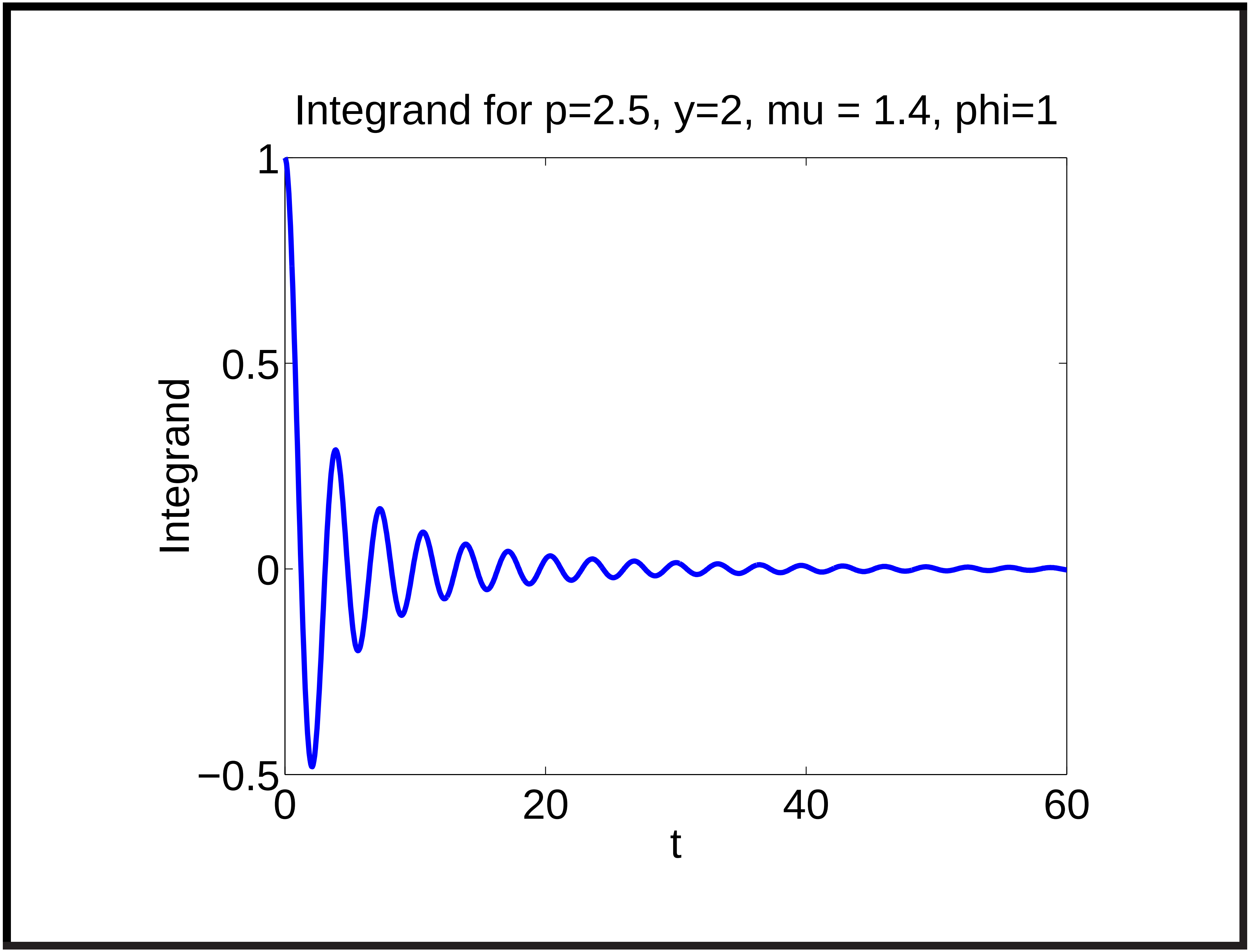
- Here, use *conditional density*, $f(y|y > 0)$.
- Then, moment generating function is

$$M_{Y|Y>0}(t) = \frac{M_Y(t) - \exp(-\lambda)}{1 - \exp(-\lambda)},$$

where $\Pr(Y = 0) = \exp(-\lambda) = \exp\left\{-\frac{\mu^{2-p}}{\phi(2-p)}\right\}$.

(NOTE: $\kappa(t) = \log M(t)$.)


Integrand for p=1.8, y=2, mu = 1.4, phi=1
Integrand
1
0.5
0
−0.5
0
20
40
60
t



Integrand for p=2.5, y=2, mu = 1.4, phi=1
Integrand
1
0.5
0
−0.5
0
20
40
60
t

Evaluation using cgf Inversion

- Integration is infinite, oscillating integral.
- Steps for inversion of cgf:
  - Determine zeros of integrand;
  - Integrate between zeros
    (use high-order Gaussian integration);
  - Sum these values until convergence.
- Speed convergence with *acceleration*.

Acceleration Methods

- Integrate between *zeros* x_j^0 of integrand giving a series of $\{A_j\}$.
- Sum should converge to the integral:

$$A = \sum_{j=1}^{\infty} A_j = \int_a^{\infty} f(x)\, dx.$$

- Speed convergence using *acceleration* methods.
- Methods due to Sidi, Aitken, Shanks, Richardson.

Sidi Acceleration

- Considers the A_i decaying as $\alpha_i \sum_i x^{-i}$.
- Implemented with W-algorithm.
- Provides an easy, fast implementation.
- Generally converges to A very quickly.

Global Approach

- Each evaluation method has strengths:

	'Small' y	**'Large'** y
$1 < p < 2$	series	inversion
$p > 2$	inversion	series

- Have estimates of when to use each method.
- Have an accurate evaluation, but can be 'slow'.
- S-PLUS functions wrapping FORTRAN code.

Fast inversion: Gridding

- *Accurately* determine some values using above.
- Store these points and *interpolate*.
- Use Chebyshev polynomials for *fast* and *accurate* interpolation.
- Use *saddlepoint approximation* for better results.

Saddlepoint Approximation

- Density: $f_p(y; \mu, \phi) = b_p(y, \phi) \exp\left\{-\frac{1}{2\phi} d(y; \mu)\right\}$.
- With $\mu = y$, gives $f_p(y; \mu = y, \phi) = b_p(y, \phi)$.
- The saddlepoint approximation is:

$$\tilde{f}_p(y; \mu, \phi) = [2\pi\phi y^p]^{-1/2} \exp\left\{-\frac{1}{2\phi} d(y; \mu)\right\}$$

- We have $f / \tilde{f} = b_p(y, \phi)\sqrt{2\pi\phi y^p} \approx 1$.

- For Tweedie densities,

$$f_p(y;\mu,\phi) = c f_p(cy; c\mu, c^{2-p}\phi).$$

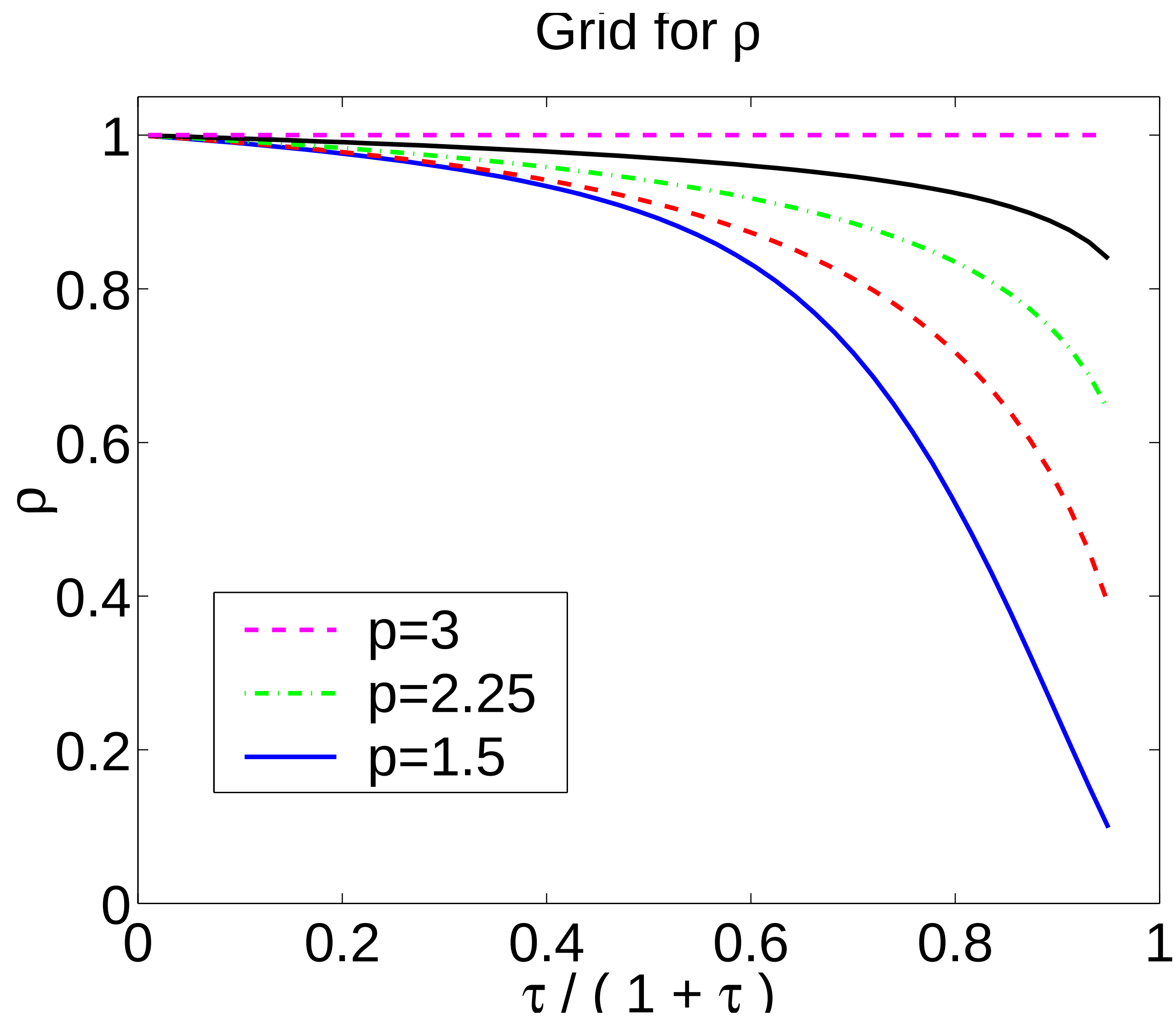
- Setting $c = 1/y$, can show that

$$f_p(y;\mu,\phi) = \frac{1}{y} b_p(1,\tau) \exp\left\{-\frac{1}{2\phi} d(y,\mu)\right\},$$

where $\tau = \phi/y^{2-p}$.

- So interpolate $\rho = \sqrt{2\pi\tau}\ b_p(1,\tau) \approx 1$.
- To restrict limits of τ, interpolate ρ over $0 \leq \tau/(1+\tau) \leq 1$.
- Reconstruct density $f_p(y;\mu,\phi)$ thus:

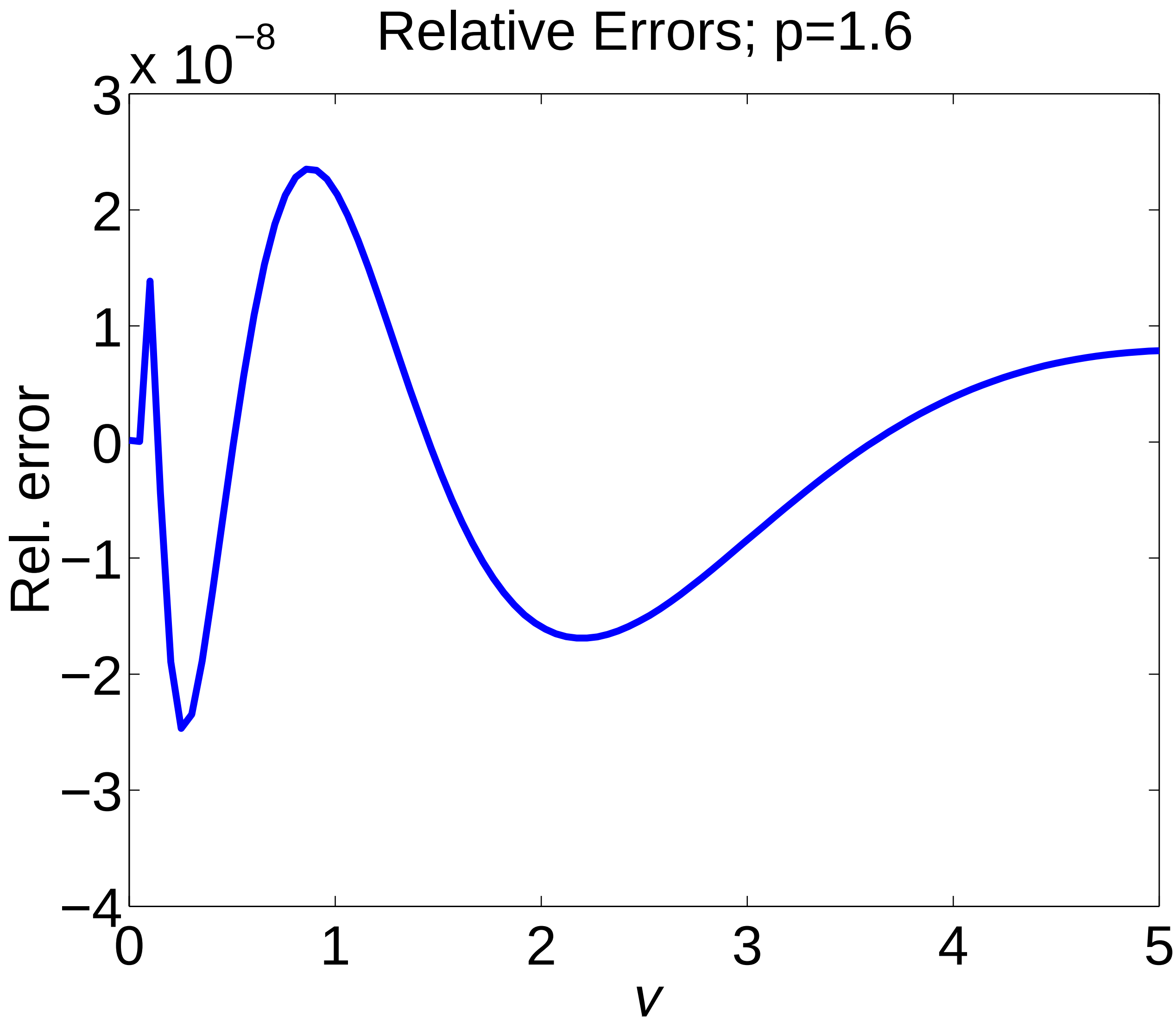
$$f_p(y;\mu,\phi) = \frac{\rho}{\sqrt{2\pi\phi y^p}} \exp\left\{-\frac{1}{2\phi} d(y;\mu)\right\}.$$

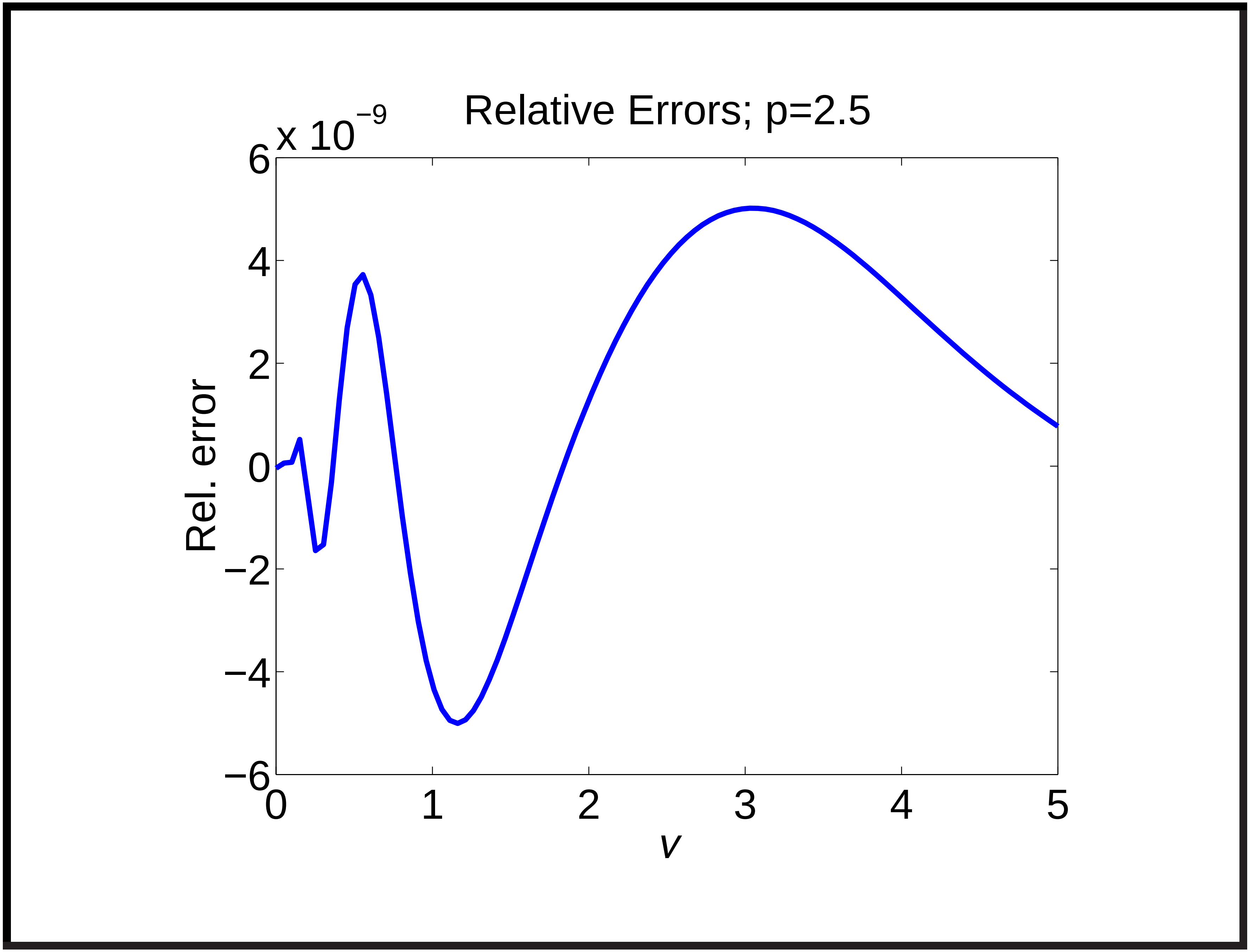


Grid for ρ
1
0.8
0.6
0.4
0.2
0
ρ
0
0.2
0.4
0.6
0.8
1
τ / (1 + τ)
p=3
p=2.25
p=1.5

Evaluation

- Evaluation using *Chebyshev polynomials*.
- Found we need to store at least 20×20 grid.
- Actual interpolation in six lines of S-PLUS code.
- Gives very accurate, very fast results.


Relative Errors; p=1.6
x 10^−8
3
2
1
0
−1
−2
−3
−4
Rel. error
0
1
2
3
4
5
v


Relative Errors; p=2.5
x 10^−9
6
4
2
0
−2
−4
−6
Rel. error
0
1
2
3
4
5
v

The Result

- S-PLUS functions available for evaluation of Tweedie densities with $p > 1$.
- Very accurate, fast answers possible.
- Now applying Tweedie family to practical problems.
- Investigations into modelling deviance.