Exchange Options under Jump-Diffusion Dynamics

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1 Introduction

- Exchange Option – exchange S_2 for S_1 at exercise time.
- Payoff at exercise time is $(S_{1,T} - S_{2,T})^+$.
- One of the simplest forms of multi-asset options.
- Applications – performance incentive fee, exchange offer, outperformance option.
- Joint work with Carl Chiarella.

- Aim of the paper.
  - $\sim$ Consider European/American style Exchange Options under jump-diffusion setting.
  - $\sim$ Impact of correlations (particularly between jump-sizes).
  - $\sim$ Representation of the option price.
  - $\sim$ Early exercise region/continuation region.
  - $\sim$ Probabilistic interpretation of the early exercise region.

2 Presentation Overview

- Review of existing results
  - $\sim$ Black-Scholes (1972) call option formula.
  - $\sim$ Merton's (1976) call option formula under jump-diffusion dynamics.
  - $\sim$ Margrabe's (1978) exchange option pricing formula.
- Radon-Nikodým derivative process with compound Poisson components for change of measure.
- Jump-diffusion setting for exchange option.
- Pricing of the European style exchange option.
- Representation of the American exchange option price.
- Probabilistic interpretation of the early exercise premium.
- Other related work on option pricing / Conclusion.

3 Brief Overview to the Option Pricing Problem

- Specify a suitable filtered probability measure space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}\}$
- Specify the dynamics for the stock price process in the form of SDE and its solution.
- If $\mathbb{P}$ is the historical measure (or market measure), a measure tranformation into a risk-neutral measure, or one of the equivalent martingale measures $\mathbb{Q}$ is needed.
  $\sim$ Associated with the EMM $\mathbb{Q}$ is a numéraire process M_t, which can be the money market account or the price process of another asset in the market.
  $\sim$ The price process of each traded asset in the market discounted by M_t is $\mathbb{Q}$-martingale.

- In order to price the option, we need to specify the form of its final pay-off e.g.
  $\sim$ Call option $(S_T - K)^+$, Put option $(K - S_T)^+$, Exchange option $(S_{1,T} - S_{2,T})^+$.
- Price of option at time t = $M_t \mathbb{E}_{\mathbb{Q}} \left[\frac{\text{final payoff}}{M_T} \middle| \mathcal{F}_t \right]$.
- If $M_t = e^{rt}$, then option price is $\mathbb{E}_{\mathbb{Q}} \left[\frac{\text{final payoff}}{e^{r(T-t)}} \middle| \mathcal{F}_t \right]$.
- If the market is complete, then there is a unique price irregardless of the numéraire and equivalent martingale measure used.

4 Review of Existing Literature

4.1 Black-Scholes' Model

- Stock price dynamics under risk-neutral measure $\mathbb{Q}$ –

$$dS_t = rS_t dt + \sigma S_t d\widetilde{B}_t.$$

- Solution to SDE is

$$S_t = S_0 \exp\left((r - 0.5\sigma^2)t + \sigma \widetilde{B}_t\right).$$

- Option pricing formula is

$$C_{BS}(S_0, K, T, r, \sigma) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

$\Phi(\ )$ is the standard normal cumulative distribution function.

4.2 Margrabe's Model

- Stock price dynamics under market measure $\mathbb{P}$ –

$$dS_{i,t} = \mu_i S_{i,t} dt + \sigma_i S_{i,t} dW_{i,t} \text{ for } i = 1, 2$$

  and $dW_{1,t}.dW_{2,t} = \rho dt$.

- Solutions to SDEs are

$$S_{i,t} = S_{i,0} \exp\left((\mu_i - 0.5\sigma_i^2)t + \sigma_i W_{i,t}\right).$$

- Exchange option pricing formula is

$$C_{Exc}(S_{1,0}, S_{2,0}, T, \sigma) = C_{BS}(S_{1,0}, S_{2,0}, T, 0, \sigma) = S_{1,0}\Phi(d_1) - S_{2,0}\Phi(d_2),$$

where

$$d_1 = \frac{\ln(S_{1,0}/S_{2,0}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

and

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.$$

4.3 Merton's Model

- Stock price dynamics under market measure is $\mathbb{P}$ –

$$\frac{dS_t}{S_{t-}} = \mu dt + \sigma dB_t + \int_{\mathbb{R}} (e^y - 1)[p(dy, dt) - \lambda m_{\mathbb{P}}(dy)dt],$$

  where jump-size distribution $y \sim N(\alpha, \delta^2)$, independent of the Wiener component, and the jump-arrival distribution.

- Stock price dynamics under Merton's risk-neutral measure is $\mathbb{Q}$ –

$$\frac{dS_t}{S_{t-}} = r dt + \sigma d\widetilde{B}_t + \int_{\mathbb{R}} (e^y - 1)[p(dy, dt) - \lambda m_{\mathbb{P}}(dy)dt].$$

- Merton assumed that jump-risk is entirely due to idiosyncratic/unsystematic risk, hence no change of distribution in the jump components under the measure transformation.

- Merton's call option formula is given by

$$C_{Mer}(S_0, K, T, r, \sigma) = \sum_{n=0}^{\infty} e^{-\hat{\lambda}T} \frac{(\hat{\lambda}T)^n}{n!} C_{BS}(S_0, K, T, r_n, \sigma_n),$$

where

$$\kappa = \mathbb{E}_{\mathbb{P}}[e^y] - 1,$$

$$\hat{\lambda} = \lambda(1+\kappa),$$

$$r_n = r - \lambda\kappa + \frac{n\alpha}{T} + \frac{n\delta^2}{2T},$$

$$\sigma_n = \sigma + \frac{n\delta^2}{T}.$$

4.4 Other Related Results

- Jamshidian (2007) – exchange option where the ratios of the two assets are exponential Poisson.
- American style single stock options under pure diffusion dynamics – Carr *et al* (1992), Jacka (1991) and Kim (1990)
- American exchange option under pure-diffusion dynamics – Bjerksund and Stensland (1993), Broadie and Detemple (1997).
- Pham (1997) and Gukhal (2001) derive a linked system of integral equations for the price and free boundary of American calls and puts (on a single stock).
- Chesney and Jeanblanc (2004) consider the case of American calls and puts in an exponential Lévy setting.

5 The Extended Model of Merton and Margrabe

- Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}\}$ be a probability measure space.
- Only interested in the filtration over $0 \leq t \leq T$ for some fixed T, the expiry time of the option, that is $\{\mathcal{F}_t\}_{t \leq T}$.
- Let $\boldsymbol{W}_t = (W_{1,t}, W_{2,t})$ be a bivariate correlated Brownian motion process which is adapted to the filtration.
- Define the matrix

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \tag{1}$$

where $dW_{1,t}.dW_{2,t} = \rho dt$, with ρ the instantaneous correlation between the two Brownian motion components.

- Also adapted to the filtration is a homogeneous Poisson counting measure $p(dy, dt)$ defined over $\mathbb{R}^2 \times [0, T]$ which is associated with a marked point process $((\boldsymbol{Y}_n), N_t)$.

- The intensity of the Poisson measure $p(dy, dt)$ is $\lambda m_{\mathbb{P}}(dy)dt$, where $\lambda > 0$ is the constant arrival rate of the jumps of the Poisson process N_t under $\mathbb{P}$ and $m_{\mathbb{P}}(dy)$ is the probability distribution on the independently and identically distributed marks $\boldsymbol{Y}_n$, also independent of N_t and $\boldsymbol{W}_t$.

- Denote its compensated measure as

$$\hat{p}(dy, dt) = p(dy, dt) - \lambda m_{\mathbb{P}}(dy)dt. \tag{2}$$

- Since we have two stocks in our model, the jump-size $\boldsymbol{Y} = (Y_1, Y_2)^\top$ is a bivariate process taking values $y = (y_1, y_2)^\top$ in $\mathbb{R}^2$.

- In our extended model, we assume that $S_{1,t}$ and $S_{2,t}$ are traded financial assets with return dynamics under the market measure $\mathbb{P}$ given by

$$\frac{dS_{i,t}}{S_{i,t-}} = \mu_i dt + \sigma_i dW_{i,t} + \int_{\mathbb{R}^2} [e^{y_i} - 1]\hat{p}(dy, dt) \tag{3}$$

for $i = 1, 2$, and where for asset i, μ_i is the instantaneous expected return per unit time, and σ_i is the instantaneous volatility per unit time.

- The Y_i $(i = 1, 2)$ are random jump-sizes assumed to be correlated pair-wise with correlation $\mathrm{Corr}[Y_1, Y_2] = \rho_{\boldsymbol{Y}}$, and we define the expected proportional jump-size

$$\kappa_i \equiv \mathbb{E}_{\mathbb{P}}\left[e^{Y_i} - 1\right] = \int_{\mathbb{R}} [e^{y_i} - 1]\, m_{\mathbb{P}}(dy_i). \tag{4}$$

- The moment generating function of all the jump-sizes $\boldsymbol{Y} = (Y_1, Y_2)^\top$ is given by

$$M_{\mathbb{P},\boldsymbol{Y}}(\boldsymbol{u}) = \mathbb{E}_{\mathbb{P}}\left[e^{\boldsymbol{u}^\top \boldsymbol{Y}}\right]. \tag{5}$$

- For each asset, the nth jumps $Y_{1,n}$ and $Y_{2,n}$ occur together, driven by the same Poisson arrival process N_t.
- The correlation between the nth return jump-sizes of each asset is determined by the moment generating function of the joint jump-size distribution (5).
- The jumps can be related to macro-economic shocks to the system.
- Should the outcome of one jump-size be zero, and the other non-zero, for example, if $Y_{1,n} = 0$ and $Y_{2,n} \neq 0$, then it could be attributed to idiosyncratic shocks.

- Note that the dynamics of the stock prices can be expressed in terms of the drift less the compensators as

$$\frac{dS_{i,t}}{S_{i,t-}} = (\mu_i - \lambda\kappa_i)dt + \sigma_i dW_{i,t} + \int_{\mathbb{R}} [e^{y_i} - 1]p(dy_i, dt), \qquad (6)$$

yielding a solution of the form

$$S_{i,t} = S_{i,0} \exp\left[\left(\mu_i - \lambda\kappa_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_{i,t} + \sum_{n=1}^{N_t} Y_{i,n}\right], \qquad (7)$$

where the jump-sizes $Y_{i,n}$ are independently and identically distributed with density $m_{\mathbb{P}}(dy_i)$.

- We will also assume that asset i pays a continuous dividend yield at rate ξ_i.
- We also assume that neither of the correlations ρ nor $\rho_{\boldsymbol{Y}}$ are ± 1.
- For now, we make no assumptions about the distribution of $\boldsymbol{Y}$. We only require its joint moment generating function (5) to exist and that its joint density $m_{\mathbb{P}}(dy)$ is non-atomic.

6 Transformation of Measures

- In our model given by (3), randomness in the model is driven by the correlated bivariate Wiener process $\boldsymbol{W}_t$ and the multivariate compound Poisson process

$$\sum_{n=1}^{N_t} \boldsymbol{Y}_n = \begin{pmatrix} \sum_{n=1}^{N_t} Y_{1,n} \\ \sum_{n=1}^{N_t} Y_{2,n} \end{pmatrix} \tag{8}$$

  where $Y_{i,n}$ is a jump size that has an identical distribution to Y_i ($i = 1, 2$) and $Y_{1,n}$ is correlated to $Y_{2,n}$ in the measure $\mathbb{P}$ as determined by the moment generating function (5).

- A suitable form for the Radon-Nikodým derivative that induces a measure transformation from the original historical measure $\mathbb{P}$ to some equivalent risk-neutral measure $\mathbb{Q}$, has to be a function of $\boldsymbol{W}_t$ as well as the compound Poisson process (8).

- Following Runggaldier (2003), let

$$\begin{aligned} L_t &= \exp\left[-(\Sigma^{-1}\boldsymbol{\theta})^\top \boldsymbol{W}_t - \frac{1}{2}\boldsymbol{\theta}^\top \Sigma^{-1}\boldsymbol{\theta} t\right] \\ &\quad \times \exp\left[-\lambda\kappa' t + \sum_{n=1}^{N_t}\left(\boldsymbol{\gamma}^\top \boldsymbol{Y}_n + \nu\right)\right], \end{aligned} \tag{9}$$

  where

$$\kappa' = e^{\nu} M_{\mathbb{P},\boldsymbol{Y}}(\boldsymbol{\gamma}) - 1 = e^{\nu}\mathbb{E}_{\mathbb{P}}\left[e^{\boldsymbol{\gamma}^\top \boldsymbol{Y}}\right] - 1. \tag{10}$$

- The process L_t is a Radon-Nikodým derivative parametrized by $\boldsymbol{\theta}$, $\boldsymbol{\gamma}$ and ν (to be interpreted below) if $\mathbb{E}_{\mathbb{P}}[L_t] = 1$ for $0 \leq t \leq T$.

- More general representations expressed in terms of Lévy measures, jump measures or compensated jump measures can be found in Colwell and Elliott (1993), Cont and Tankov (2004a) and Runggaldier (2003).

- The following theorem which we use is standard.

Theorem 1. *Consider the probability measure space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ *over the time interval* $[0, T]$ *such that the Wiener components* $\boldsymbol{W}_t = (W_{1,t}, W_{2,t})$ *and the compound Poisson process* $\sum_{n=0}^{N_t} \boldsymbol{Y}_n$ *are adapted to the filtration* $\mathcal{F}_t$*. Suppose* L_t *given by (9) and that* $\mathbb{E}_{\mathbb{P}}[L_T] = 1$*. Then* L_t *is a Radon-Nikodým derivative of some equivalent measure* $\mathbb{Q}$ *with respect to* $\mathbb{P}$*, that is,* $L_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_t$*. Then the Wiener components* $W_{i,t}$ *have drift* $-\theta_i$ *under the measure* $\mathbb{Q}$ *and the compound Poisson process* $\sum_{n=0}^{N_t} \boldsymbol{Y}_n$ *has a new intensity rate*

$$\tilde{\lambda} = \lambda(1 + \kappa') \tag{11}$$

and a new distribution for the jump sizes given by the moment generating function

$$M_{\mathbb{Q},\boldsymbol{Y}}(\boldsymbol{u}) = \frac{M_{\mathbb{P},\boldsymbol{Y}}(\boldsymbol{u} + \boldsymbol{\gamma})}{M_{\mathbb{P},\boldsymbol{Y}}(\boldsymbol{\gamma})}. \tag{12}$$

- The non-uniqueness of the measure $\mathbb{Q}$ stems from the arbitrariness of the parameters $\boldsymbol{\gamma}$ and ν.
- Some specific choices of the new measure $\mathbb{Q}$ can be determined by specific values of $\boldsymbol{\gamma}$ and ν in the Radon-Nikodým derivative (9).
- The choice of $\boldsymbol{\gamma} = \mathbf{0}$ and $\nu = 0$ is analagous to the case in Merton's (1976) jump-diffusion model where the jump-risk is unpriced.
- If $\boldsymbol{\gamma} = \mathbf{0}$ but $\nu \neq 0$, then there is a change to the jump-arrival intensity but not to the jump-size distribution under the measure transformation.
- Lastly, if $\nu = -\ln M_{\mathbb{P},\boldsymbol{Y}}(\boldsymbol{\gamma})$, then the jump-arrival intensity does not change although the distribution of the jump-sizes change under the transformation of measure.
- The values of θ_i are determined by the martingale condition on the discounted stock yield processes only after $\boldsymbol{\gamma}$ and ν have been chosen.

- The Wiener part of the Radon-Nikodým derivative, is the usual Radon-Nikodým derivative for correlated Wiener processes and via the usual Girsanov's theorem, there exists $\widetilde{W}_t = (\widetilde{W}_{1,t}, \widetilde{W}_{2,t})$, correlated standard Brownian motion components under the transformed measure $\mathbb{Q}$ such that

$$d\widetilde{W}_{i,t} = \theta_i dt + dW_{i,t}, \tag{13}$$

for $i = 1, 2$ and $d\widetilde{W}_{1,t}.d\widetilde{W}_{2,t} = \rho dt$.

- The jump part of the Radon-Nikodým derivative (9) can be written in a more general form involving a compensated Poisson measure.

- However we wish to preserve the homogeniety of the Poisson process N_t after the measure transformation and hence the form of the Radon-Nikodým derivative (9) with constant parameters.

7 The IPDE for the Exchange Option

- Suppose that one way or another we have determined the parameters $\boldsymbol{\gamma}$ and ν and hence have a pricing measure $\mathbb{Q}$.
- Assume the existence of a money market account $M_t = e^{rt}$ where r is the constant risk-free rate.
- Under a risk-neutral measure $\mathbb{Q}$, we require the discounted yield process of each stock $\frac{S_{i,t}e^{\xi_i t}}{e^{rt}}$ to be martingales.
- After the parameters $\boldsymbol{\gamma}$ and ν in the Radon-Nikodým derivative (9) have been chosen, the Poisson counting measure $p(dy, dt)$ has intensity $\tilde{\lambda} m_{\mathbb{Q}}(dy)dt$ where $\tilde{\lambda}$ is given by (11) and the joint distribution of the jump sizes $m_{\mathbb{Q}}(dy)$ is determined by the moment generating function (12).

- Define the compensated Poisson measure under $\mathbb{Q}$ as

$$\hat{q}(dy, dt) = p(dy, dt) - \tilde{\lambda} m_{\mathbb{Q}}(dy)dt, \tag{14}$$

and the expected jump-size increment of the ith stock under $\mathbb{Q}$ as

$$\tilde{\kappa}_i = \int_{\mathbb{R}} [e^{y_i} - 1] m_{\mathbb{Q}}(dy_i) = \int_{\mathbb{R}^2} [e^{y_i} - 1] m_{\mathbb{Q}}(dy), \tag{15}$$

where $m_{\mathbb{Q}}(dy_i)$ is the marginal relative jump-size distribution for the ith asset under $\mathbb{Q}$.

- Then the dynamics for the discounted stock yield processes can be expressed as

$$d\left(\frac{S_{i,t}e^{\xi_i t}}{e^{rt}}\right) = \left(\frac{S_{i,t-}e^{\xi_i t}}{e^{rt}}\right)\left[(\mu_i + \xi - r - \lambda\kappa_i + \tilde{\lambda}\tilde{\kappa}_i)dt + \sigma_i dW_{i,t} + \int_{\mathbb{R}^2}[e^{y_i} - 1]\hat{q}(dy, dt)\right], \tag{16}$$

for $i = 1, 2$.

- Thus we can set θ_i in the Radon-Nikodým derivative (9) to be

$$\theta_i = \frac{(\mu_i + \xi_i - r - \lambda\kappa_i + \tilde{\lambda}\tilde{\kappa}_i)}{\sigma_i} \tag{17}$$

  so that

$$d\widetilde{W}_{i,t} = \frac{(\mu_i + \xi_i - r - \lambda\kappa_i + \tilde{\lambda}\tilde{\kappa}_i)}{\sigma_i} dt + dW_{i,t}, \tag{18}$$

  and $\widetilde{W}_{i,t}$ are standard Brownian motion components under $\mathbb{Q}$ and $d\widetilde{W}_{1,t}.d\widetilde{W}_{2,t} = \rho dt$.

- By rewriting the expressions for θ_i into the form

$$\mu_i + \xi_i - r = \theta_i\sigma_i + (\lambda\kappa_i + \tilde{\lambda}\tilde{\kappa}_i) = \theta_i\sigma_i + \lambda\int_{\mathbb{R}^2}[e^{y_i} - 1][1 - e^{\boldsymbol{\gamma}^\top y + \nu}]m_{\mathbb{P}}(dy), \tag{19}$$

  we see that the risk-premia of each asset may be expressed as the sum of its Wiener risk-premium and its jump-risk premium.

- The term θ_i can be interpreted as the market price of Wiener risk, and $(\lambda\kappa_i + \tilde{\lambda}\tilde{\kappa}_i) = \lambda \int_{\mathbb{R}^2}[e^{y_i} - 1][1 - e^{\boldsymbol{\gamma}^\top y+\nu}]m_{\mathbb{P}}(dy)$ as the jump-risk premium for the ith asset.
- Thus the dynamics for the discounted asset yield processes can be expressed as

$$d\left(\frac{S_{i,t}e^{\xi_i t}}{e^{rt}}\right) = \left(\frac{S_{i,t-}e^{\xi_i t}}{e^{rt}}\right)\left[\sigma_i d\widetilde{W}_{i,t} + \int_{\mathbb{R}^2}[e^{y_i} - 1]\hat{q}(dy, dt)\right], \tag{20}$$

  for $i = 1, 2$.

- From (20), the dynamics for the asset prices under $\mathbb{Q}$ is then

$$dS_{i,t} = S_{i,t-}\left[(r - \xi_i)dt + \sigma_i d\widetilde{W}_{i,t} + \int_{\mathbb{R}^2}[e^{y_i} - 1]\hat{q}(dy, dt)\right], \tag{21}$$

yielding for the asset prices the solution

$$S_{i,t} = S_{i,0} \exp \left[\left(r - \xi_i - \frac{\sigma_i^2}{2} - \tilde{\lambda}\tilde{\kappa}_i \right) t + \sigma_i \widetilde{W}_{i,t} + \sum_{n=0}^{N_t} Y_{i,n} \right]. \tag{22}$$

- Consider a European exchange option based on the two assets such that the final payoff is $(S_{1,T} - S_{2,T})^+$, that is, the holder of the option will exchange one unit of stock S_1 for one unit of stock S_2 if it is to the holder's advantage.

- Since the asset prices $(S_{1,2}, S_{2,t})$ are jointly Markov, and the final payoff of the option is not path-dependent, the European option price C_t^E at time t is a function of time t and the asset prices $S_{1,t}$ and $S_{2,t}$ at time t only, and we can write the price as $C_t^E(S_{1,t}, S_{2,t})$.

- Likewise we denote the American option price $C_t^A(S_{1,t}, S_{2,t})$.

- Thus at maturity, the European exchange (as well as the American exchange) payoff is

$$C_T^E(S_{1,T}, S_{2,T}) = C_T^A(S_{1,T}, S_{2,T}) = (S_{1,T} - S_{2,T})^+. \tag{23}$$

- Given a risk-neutral measure $\mathbb{Q}$ that is already determined by the choice of parameters in the Radon-Nikodým derivative (9), the risk-neutral valuation of the European call option price is

$$C_t^E(S_{1,t}, S_{2,t}) = \mathbb{E}_{\mathbb{Q}}\left[\left. \frac{(S_{1,T} - S_{2,T})^+}{e^{r(T-t)}} \right| \mathcal{F}_t \right] = \mathbb{E}_{\mathbb{Q}}\left[\left. \frac{(S_{1,T} - S_{2,T})^+}{e^{r(T-t)}} \right| S_{1,t}, S_{2,t} \right]. \tag{24}$$

- We state a theorem that yields the integro-partial differential equation for the European option price.

Theorem 2. *The European option price $C_t^E(S_{1,t}, S_{2,t})$ given by (24) with (22) as the underlying asset price dynamics, satisfies the integro-partial differential equation*

$$\mathcal{L}C_{t-}^E(S_{1,t-}, S_{2,t-}) + \int_{\mathbb{R}^2} \left[C_t^E(S_{1,t-}e^{y_1}, S_{2,t-}e^{y_2}) - C_t^E(S_{1,t-}, S_{2,t-}) \right] m_{\mathbb{Q}}(dy)$$
$$= rC_t^E(S_{1,t-}, S_{2,t-}), \quad (25)$$

subject to the final boundary condition $C_T^E(S_{1,T}, S_{2,T}) = (S_{1,T} - S_{2,T})^+$, where the operator $\mathcal{L}$ is defined by

$$\mathcal{L} = \partial_t + \sum_{i=1}^{2} (r - \xi_i - \tilde{\lambda}\tilde{\kappa}_i) S_{i,t-}\partial_i + \frac{\sigma_1^2}{2} S_{1,t-}^2 \partial_{11} + \frac{\sigma_2^2}{2} S_{2,t-}^2 \partial_{22} + \rho\sigma_1\sigma_2 S_{1,t-} S_{2,t-} \partial_{12}. \quad (26)$$

8 Pricing the European Exchange Option

- Here we derive a formula (in the form of a suitable Poisson average) for the European exchange option when the asset prices follow the jump-diffusion dynamics (3).

- We further assume that under the original historical measure $\mathbb{P}$, the jump-sizes $\boldsymbol{Y}_n$ are independently and identically distributed as multivariate normal $MVN(\boldsymbol{\alpha}, \Sigma_{\boldsymbol{Y}})$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^\top$ and

$$\Sigma_{\boldsymbol{Y}} = \begin{pmatrix} \delta_1^2 & \rho_{\boldsymbol{Y}}\delta_1\delta_2 \\ \rho_{\boldsymbol{Y}}\delta_1\delta_2 & \delta_2^2 \end{pmatrix}. \tag{27}$$

- Under $\mathbb{Q}$, the jump-sizes $\boldsymbol{Y}_n$ are independently and identically distributed as multivariate normal $MVN(\tilde{\boldsymbol{\alpha}}, \Sigma_{\boldsymbol{Y}})$, where $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1, \tilde{\alpha}_2)^\top$.

- From the application of (12) in Theorem 1, we find that $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} + \Sigma_{\boldsymbol{Y}}\boldsymbol{\gamma}$, that is, $\tilde{\alpha}_1 = \alpha_1 + \gamma_1\delta_1^2 + \gamma_2\rho_{\boldsymbol{Y}}\delta_1\delta_2$ and $\tilde{\alpha}_2 = \alpha_2 + \gamma_2\delta_2^2 + \gamma_1\rho_{\boldsymbol{Y}}\delta_1\delta_2$.
- We price the exchange option by using the money market account as the numéraire, in contrast to the approach in Cheang, Chiarella and Ziogas (2006) where Margrabe's original approach is followed and one of the asset yield processes is the numéraire instead.

Theorem 3. *Suppose the asset prices follow the dynamics (21), and the continuous dividend rate for each asset is ξ_i. Then when $S_{1,t} = s_1$ and $S_{2,t} = s_2$, the European exchange option price is*

$$\begin{aligned} C_t^E(s_1, s_2) &= \sum_{n=0}^{\infty} e^{-\tilde{\lambda}(T-t)} \frac{(\tilde{\lambda}(T-t))^n}{n!} \\ &\quad \times \Big[s_1 e^{-(\xi_1 + \tilde{\lambda}\tilde{\kappa}_1)(T-t) + n\tilde{\alpha}_1 + \frac{n\delta_1^2}{2}} \Phi(d_{1,n,t}) \\ &\quad - s_2 e^{-(\xi_2 + \tilde{\lambda}\tilde{\kappa}_2)(T-t) + n\tilde{\alpha}_2 + \frac{n\delta_2^2}{2}} \Phi(d_{2,n,t}) \Big], \end{aligned} \tag{28}$$

where

$$d_{1,n,t} = \frac{\ln\frac{s_1}{s_2} + (\xi_2 - \xi_1 - \tilde{\lambda}(\tilde{\kappa}_1 - \tilde{\kappa}_2) + \sigma^2/2)(T-t) + n(\tilde{\alpha}_1 - \tilde{\alpha}_2 + \delta^2/2)}{\sigma_n\sqrt{T-t}},$$

and

$$d_{2,n,t} = d_{1,n,t} - \sigma_n\sqrt{T-t};$$

with

$$\delta^2 = \delta_1^2 + \delta_2^2 - \rho_{\mathbf{Y}}\delta_1\delta_2,$$

and

$$\sigma_n^2 = \sigma^2 + \frac{n\delta^2}{T}$$

with

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2,$$

where Φ *is the standard normal probability distribution function.*

- The proof is based on the expansion of the conditional expectation

$$\mathbb{E}_{\mathbb{Q}}\left[\left.\frac{(S_{1,T}-S_{2,T})^{+}}{e^{r(T-t)}}\right|\mathcal{F}_t\right].$$

- In the proof, change of numéraire techniques in the manner of Geman *et al* (1995) and Eberlein *et al* (2009), in conjuction with Theorem 1, are used.

- Note that if in (28) we set $\tilde{\alpha}_i = \delta_i = 0$ (and consequently $\tilde{\kappa}_i = 0$), then we would obtain the pricing formula for Margrabe's (1978) exchange option model under pure-diffusion dynamics.

9 A Representation for the American Exchange Option Price

- Now we turn our attention to the pricing of the American exchange option.
- We assume that a risk-neutral measure $\mathbb{Q}$ has already been chosen from an a priori choice of the parameters γ and ν in Theorem 1.
- Since the European exchange price is taken to be the risk-neutral valuation of the discounted final payoff, then from the usual argument using the Snell envelope, the discounted American exchange price C_t^A at time t is then taken to be the Snell envelope of the discounted intrinsic value $e^{-rt}(S_{1,t} - S_{2,t})^+$, thus

$$\frac{C_t^A(S_{1,t}, S_{2,t})}{e^{rt}} = \text{ess} \sup_{\tau \in [t,T]} \mathbb{E}_{\mathbb{Q}} \left[\frac{(S_{1,\tau} - S_{2,\tau})^+}{e^{r\tau}} \middle| \mathcal{F}_t \right]. \tag{29}$$

- Since the two asset price processes are jointly Markov, the American exchange option price is given by

$$\begin{aligned} C_t^A(S_{1,t}, S_{2,t}) &= \operatorname*{ess\,sup}_{\tau \in [t,T]} \mathbb{E}_{\mathbb{Q}} \left[\frac{(S_{1,\tau} - S_{2,\tau})^+}{e^{r(\tau - t)}} \middle| \mathcal{F}_t \right] \\ &= \operatorname*{ess\,sup}_{\tau \in [t,T]} \mathbb{E}_{\mathbb{Q}} \left[\frac{(S_{1,\tau} - S_{2,\tau})^+}{e^{r(\tau - t)}} \middle| S_{1,t}, S_{2,t} \right]. \quad (30) \end{aligned}$$

- A quantity of interest in American option pricing is the early exercise boundary which in the current context will depend on the time t.

- Following Broadie and Detemple (1997), the American exchange option is exercised at time t when $S_{1,t} \geq b_t S_{2,t}$, where $s_{1,t} = b_t s_{2,t}$ denotes the early exercise line at time t (see Fig. 1).

- Fig. 1 demonstrates the continuation and stopping regions, and early exercise boundary for the American exchange option under consideration.

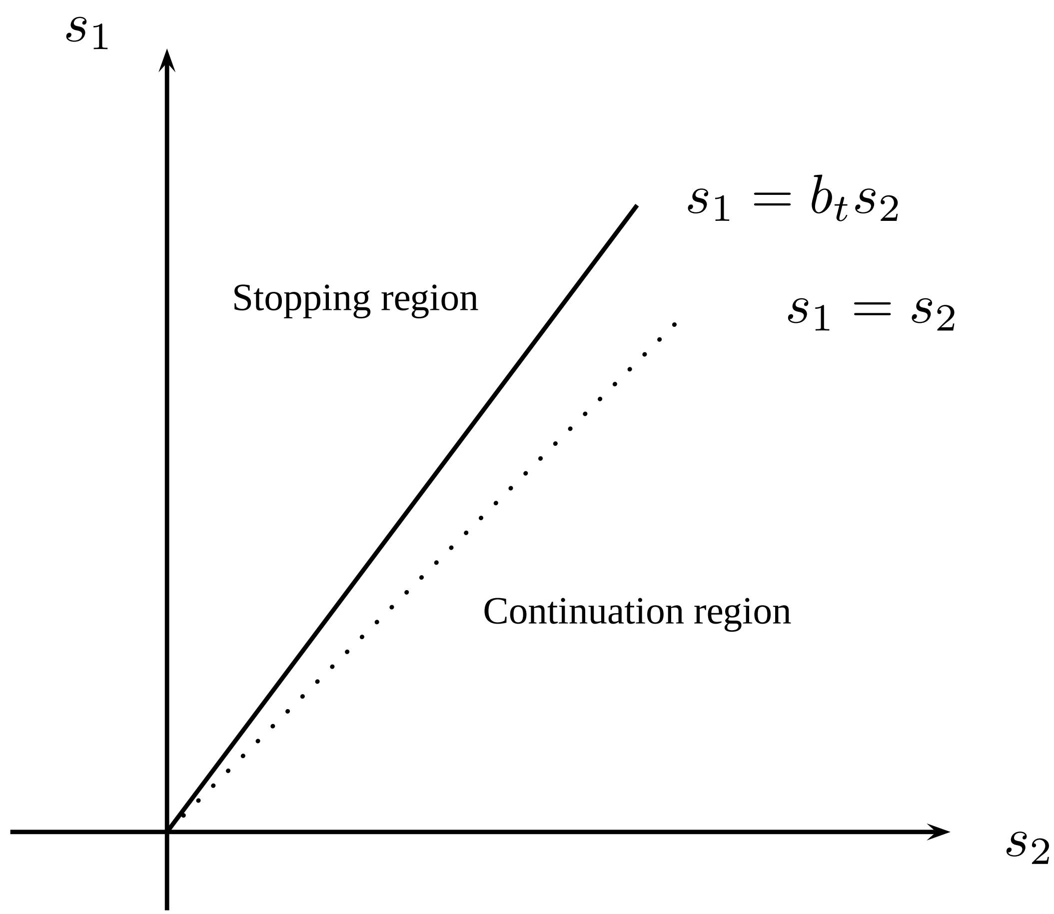

Figure 1: Continuation and stopping regions for the American exchange option, at a given time t. The early exercise boundary is the line $s_1 = b_t s_2$.

Theorem 4. *The American exchange option value decomposes as*

$$C_t^A(S_{1,t},S_{2,t}) = C_t^E(S_{1,t},S_{2,t}) + C_t^P(S_{1,t},S_{2,t}) \tag{31}$$

where the first term on the right hand side is the European exchange option price and the second term is the early exercise premium. The early exercise premium term can be written as

$$C_t^P(S_{1,t},S_{2,t}) = C_t^{P,D}(S_{1,t},S_{2,t}) - \tilde{\lambda} C_t^{P,J}(S_{1,t},S_{2,t}), \tag{32}$$

where

$$C_t^{P,D}(S_{1,t},S_{2,t}) = \int_t^T e^{-r(\tau-t)}\mathbb{E}_{\mathbb{Q}}\left[\left(\xi_1 S_{1,\tau-} - \xi_2 S_{2,\tau-}\right)\mathbb{1}_{\{S_{1,\tau-}\geq b_\tau S_{2,\tau-}\}}\middle|\mathcal{F}_t\right]d\tau \tag{33}$$

and

$$C_t^{P,J}(S_{1,t}, S_{2,t}) = \int_t^T e^{-r(\tau-t)}$$

$$\times\, \mathbb{E}_{\mathbb{Q}}\left[\int_{\mathbb{R}^2} \left[C_\tau^A(S_{1,\tau-}e^{y_1}, S_{2,\tau-}e^{y_2}) - (S_{1,\tau-}e^{y_1} - S_{2,\tau-}e^{y_2}) \right] \mathbb{1}_{\widetilde{B}(\tau)} m_{\mathbb{Q}}(dy) \middle| \mathcal{F}_t \right] d\tau, \tag{34}$$

where the event $\widetilde{B}(\tau)$ *is defined by*

$$\widetilde{B}(\tau) = \left\{ b_\tau \le \frac{S_{1,\tau-}}{S_{2,\tau-}} < b_\tau e^{y_2-y_1} \right\}$$

with b_τ *being the early exercise boundary at time* τ*.*

- Proof is based on the observation that the American option price satisfies the following boundary and smooth-pasting conditions, given the exercise boundary b_t.

$$C_t^A(S_{1,t}, S_{2,t}) = S_{1,t} - S_{2,t} \quad \text{where} \quad S_{1,t} \geq b_t S_{2,t}, \tag{35}$$

$$\lim_{(s_1,s_2)\to(s_1,s_1/b_t)} \frac{\partial C_t^A(s_1, s_2)}{\partial s_1} = 1, \tag{36}$$

$$\lim_{(s_1,s_2)\to(b_t s_2,s_2)} \frac{\partial C_t^A(s_1, s_2)}{\partial s_2} = -1. \tag{37}$$

- In addition, the condition

$$\frac{\partial C_t^A(s_1, s_2)}{\partial t} = 0, \quad \text{where} \quad s_1 > b_t s_2 \tag{38}$$

must also be satisfied.

- Each component of the early exercise premium in (32) has a distinct financial interpretation.
- The first part, $C_t^{P,D}(S_{1,t}, S_{2,t})$ given by (33), denotes the component of the early exercise premium arising from the diffusion part of the dynamics for the stock.
- Specifically, $C_t^{P,D}(S_{1,t}, S_{2,t})$ is the expected present value of the portfolio $\xi_1 S_1 - \xi_2 S_2$ held whenever the combination of the stock prices (S_1, S_2) is in the stopping region.
- The second term, $\tilde{\lambda} C_t^{P,J}(S_{1,t}, S_{2,t})$ as given in (34), arises from the presence of jumps, and is the expected present value of the cost incurred by the option holder whenever the combination of the stock prices (S_1, S_2) jumps from the stopping region back into the continuation region.

- This is the "rebalancing cost due to jumps from the exercise region into the continuation region" analagous to the cases for call and put options for a single stock case in Gukhal (2001).
- As the combination of stock prices jump from the exercise region into the continuation region, the investor does not get a chance to exercise the option since the jumps are instantaneous.
- So the investor ends up paying the difference between the American option and its exercise value and the present value of the expected cost, over the life of the option, of these jumps back into the continuation region.

- The event $\widetilde{B}(\tau)$ is basically the event that the combination of the stock prices (S_1, S_2) at time just prior to τ were in the stopping region but jumps at time τ put the combination of the stock prices back into the continuation region.
- Note that when there are no jumps in the model, $\tilde{\lambda} = 0$ and $C_t^{P,D}(S_{1,t}, S_{2,t})$ no longer contributes towards the early exercise premium, thus yielding the Broadie and Detemple (1997) decomposition for the American exchange option price under pure diffusion dynamics.

- There are some interesting consequences of Theorem 4.
- In the pure-diffusion case with no dividends, the value of the early-exercise premium in (32) is zero.
- This corresponds to Margrabe's (1978) observation that the American exchange option under pure-diffusion dynamics should be optimally exercised at maturity.
- In the case where we have pure-diffusion dynamics with dividends but no jumps, we obtain the Broadie and Detemple (1997) result.

- Suppose the asset prices follow the jump-diffusion dynamics given by (3) but do not pay dividends, then the early exercise premium (32) consists of the term $-\tilde{\lambda} C_t^{P,J}(S_{1,t}, S_{2,t})$ only.

- In the event that the asset $S_{1,t}$ is allowed to have positively valued jumps and asset $S_{2,t}$ allowed to have only negatively valued jumps, then the term $C_t^{P,J}(S_{1,t}, S_{2,t})$ in the early exercise premium (32) is zero since the event $\widetilde{B}(\tau)$ is an empty set, which is similar to the cases in Chesney and Jeanblanc (2004) where they considered cases for American calls and puts on an asset with the appropriate single-sided jumps.

- In order to evaluate the option price from equation (31) we need an expression for the early exercise boundary b_t.
- Using the fact that at the early exercise boundary we have

$$C_t^A(S_{1,t}, S_{1,t}/b_t) = S_{1,t}(1 - 1/b_t), \tag{39}$$

or equivalently

$$C_t^A(b_t S_{2,t}, S_{2,t}) = S_{2,t}(b_t - 1), \tag{40}$$

we can obtain the integral equation that determines the free boundary (see Corollary 1)

Corollary 1. *The free boundary b_t is determined by the integral equation*

$$S_{1,t}(1 - 1/b_t) = C_t^E(S_{1,t}, S_{1,t}/b_t) + C_t^P(S_{1,t}, S_{1,t}/b_t), \tag{41}$$

or equivalently

$$S_{2,t}(b_t - 1) = C_t^E(b_t S_{2,t}, S_{2,t}) + C_t^P(b_t S_{2,t}, S_{2,t}), \tag{42}$$

- It should be kept in mind that the term $C_t^P(S_{1,t}, S_{1,t}/b_t)$ or $C_t^P(b_t S_{2,t}, S_{1,t})$ involves the unknown option price though its second term (the one arising due to the presence of jumps).
- Hence (31) and (41) (or (42)) need to be solved as a linked system, this is in contrast to the corresponding system of equations in the no jump case which can be solved sequentially, first the integral equation for the free boundary and then use that to evaluate the option price.
- Note that in (41) or in (42), b_t can be solved numerically given $S_{1,t}$ or $S_{2,t}$ only.

10 Other Related Work

- Geman *et al* (1995) have provided a call option pricing formula with a stochastic bond price under pure diffusion dynamics.
- Since $P_{T,T} = 1$, the final payoff of the call option is

$$(S_T - K)^+ = (S_T - KP_{T,T})^+.$$

- Cheang and Teh (2008) extended Geman *et al* (1995) to the case where both the stock and the bond prices are modelled by Geometric Jump-Diffusion processes.

- We acknowledge the suggestion from Yoshio Miyahara regarding the possibility of extending Geman *et al* (1995) and Cheang and Teh (2008) results to a more general Geometric Lévy process setting.
- Carl Chiarella has suggested a possible extension of Cheang and Teh (2008) to the American style option case with a dividend paying stock.

11 Conclusion

- Extended the results of Merton (1976) and Margrabe (1978) to exchange options under jump-diffusion dynamics.
- Considered both the European and American exchange option problem under this setting.
- Priced the European exchange option under a family of equivalent martingale measures.
- Shown how the non-uniqueness of the option price manifests itself through variations of the parameters in the Radon-Nikodým derivative.
- Extended the Broadie and Detemple (1997) results from the pure diffusion to the jump-diffusion setting.

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