MAT3106

Vector Calculus and Mathematical Modelling of Fluid Flows

Faculty of Sciences

Study Book

Written by

Tony Roberts & Sergey Suslov
Department of Mathematics & Computing
Faculty of Sciences
The University of Southern Queensland

Distributed by

The University of Southern Queensland
Toowoomba, Queensland 4350
Australia
http://www.usq.edu.au

Produced using LaTeX by the Dept of Maths & Computing, http://www.sci.usq.edu.au, in a style with minor adaptations from a base of the refrep.cls (v2.0d) by Kielhorn & Partl, to implement Wendy Priestly's *Instructional typographies using desktop publishing techniques to produce effective learning and training materials,* http://www.ascilite.org.au/ajet/ajet7/priestly.html.

Preface

This course brings in an emphasis upon developing applications, mainly fluid dynamics, side by side with the development of mathematical concepts and techniques. Some parts of the course are in the mainstream, and other parts are peripheral for a richer picture. As you read the study book you will see that we have endeavoured to convey the importance of the various concepts and sections. For example, concepts and formulae in the "aims" and the Summaries are the most essential. In support of this, the reading you have been asked to do has been classified by requests to "study", "read" or "peruse" in order of decreasing importance. To direct your study a number of specific problems and questions are suggested in the study book that you have to solve and submit the answers to the lecturer for feedback. These problems and questions (listed in the Activity sections of the study book) are a minimum that you should be able to do immediately. They contribute to your weekly homework which is a part of your assessment (see course specifications for detail). You are expected to contact the lecturer with any difficulties you have answering the homework questions and solving the problems before the corresponding homework submission deadlines. Remember that these regular weekly discussions with the lecturer will provide you with sufficient individual hints to solve correctly all homework problems. Therefore any student is guaranteed to receive full marks for the homework provided he/she initiates and maintains regular contact with the lecturer and observes all submission deadlines throughout the semester. Do not neglect this unique opportunity to boost your final grade! Although the lecturer might allow a late submission of homework items in exceptional cases, no discussions will be held after the regular homework submission deadlines has passed. Therefore it is extremely important that you pace yourself through the course material in strict accordance with the course schedule given in the introductory book. It is also essential for students who cannot be present on campus on a regular basis to have reliable regular access to email and/or fax and phone for individual weekly communications with the lecturer.

Sprinkled within the first part of this study guide are MATLAB scripts to enhance your ability to probe the problems and concepts and thus to improve learning. While purchasing a MATLAB license is not essential for this course, it will be beneficial if you have access to this or similar software which will enable you to visualise what mathematical contents of this course predict. Access to computer algebra software such as MAPLE is also beneficial, especially for the second half of the course, but is not a requirement.

Many of the Activity sections in this study book recommend watching the video recordings relevant to the material. The references provided point to the original tapes available from the USQ library, but for your convenience a CD-ROM [CD08] has been produced and included in your study package which includes most of the relevant clips. Although the original videotapes contain additional illustrative footage which can help you to see the developed mathematical concepts in action these are not crucial for your overall success in the course.

Finally, the teaching staff works constantly on improving the quality of this study book. Your comments, positive or negative, including reports of errors you might find in the text will be appreciated. Any essential course material updates will be published immediately on the course website at http://www.sci.usq.edu.au/courses/mat3106. You should check it regularly.

Table of Contents

Bibliography

[BE91] *Bernoulli's Equation. Video recording.* BBC TV Production for the Open University. London, UK. USQ library catalogue number 532.5 BER, 1991. 58

[CD08] *MAT3106 CD-ROM, Version 2.0.* USQ Toowoomba, 2008. 4, 37, 41, 53, 58, 97, 99, 101, 103, 120, 126, 131

[Dyk82] M. Van Dyke. *An Album of Fluid Motion.* The Parabolic Press, Stanford, California, U.S.A. USQ library catalogue number 532.05 Alb, 1982. 101, 117, 119

[FDO] *Fluid Dynamics of Drag: Part II. Video recording.* Encyclopedia Britannica Educational Corporation, Castle Hill, NSW, Australia. USQ library catalogue number 532.0533 FLU. 37, 53, 97, 99, 103

[FF85] *Fluid Flows (Engineering Mechanics: solids and fluids). Video recording.* BBC TV Production for the Open University. London, UK. USQ library catalogue number 620.1 ENG, 1985. 56, 101, 120

[FOB] *Fundamentals of Boundary Layers. Video recording.* Encyclopedia Britannica Educational Corporation, Castle Hill, NSW, Australia. USQ library catalogue number 532 FUN. 120

[FV] *Flow Visualisation. Video recording.* Encyclopedia Britannica Educational Corporation, Castle Hill, NSW, Australia. USQ library catalogue number 532.051 FLO. 41

[Kre06] E. Kreyszig. *Advanced engineering mathematics.* Wiley, 9th edition, 2006. 11, 12, 13, 20, 22, 25, 31, 32, 35, 37, 38, 42, 50, 51, 52, 64, 65, 70, 71, 72, 73, 74, 76, 78, 79, 80, 83, 85, 87, 88, 99, 100, 102

[LRN] *Low Reynolds Number Flow. Video recording.* Encyclopedia Britannica Educational Corporation, Castle Hill, NSW, Australia. USQ library catalogue number 532.051 LOW. 126, 131

[MAT] *MAT2100 Algebra and Calculus II. Study book.* USQ, Toowoomba. 97

[OO95] H. Ockendon and J.R. Ockendon. *Viscous flow.* Cambridge University Press, Cambridge, UK, USQ library catalogue number 532.0533 Ock, 1995. 96, 97, 98, 99, 102, 104, 105, 111, 116, 119, 121, 125, 127, 129, 132

[PF] *Pressure Fields and Fluid Acceleration. Video recording.* Encyclopedia Britannica Educational Corporation, Castle Hill, NSW, Australia. USQ library catalogue number 532.56 PRE. 58

[Sch79] H. Schlichting. *Boundary-layer theory.* McGraw-Hill, New York, U.S.A. USQ library catalogue number 532.052 Sch., 7th edition, 1979. 104, 107, 116, 121, 125, 127

[Whi94] F. M. White. *Fluid Mechanics.* Mc Graw-Hill, New York, U.S.A. USQ library catalogue number 620.106 Whi, 3rd edition, 1994. 107, 116, 128

I. Vector Calculus and Flow of Inviscid Fluid

Module 1 Modelling Fluid Flow Needs Vector Differentiation

We live in a constantly moving world. Light breeze and violent tornadoes, calm river flow and devastating tsunami are just a few examples of this motion. Despite their enormous variety these phenomena have at least two common features. They all are caused by the motion of continuum media such as air or water and they all can be described by the same mathematical model — a system of Navier-Stokes equations. Naturally, the model which describes such a wide variety of flows cannot be simple. You will see it for yourself in the second part of this course. Here we start with introducing a set of mathematical concepts and operators which are necessary to build a comprehensive model of fluid dynamics. In this first module we investigate the various sorts of derivatives and their application to fluid flows.

The concepts and operations of vector calculus, *grad*, *div* and *curl*, are used in many disciplines such as electromagnetism, gravity, quantum physics, mathematical biology, etc. However, here we mostly confine application to fluids: either gases or liquids such as air or water. The textbook by Kreyszig [Kre06] referenced in this study book develops the mathematical techniques; Part I of the study book supplements the text by developing the application to fluid dynamics.

Module contents

Some material usefully supporting the concepts developed in this module is to be found on the World Wide Web (WWW). For example, at the time of writing the Connected Curriculum Project has material which you may like to explore. It can be found at `http://www.math.montana.edu/frankw/ccp/multiworld/topic.htm`.

Reading 1.A → Peruse §9.1–3 in Kreyszig [Kre06, pp. 364–384]. You must be totally comfortable with this elementary material.

Also peruse §9.5–9.6 in Kreyszig [Kre06, pp. 389–403] to refresh material studied in Algebra & Calculus II. Do some of the relevant problems to remind yourself of the practice.

1.1 Scalar and vector fields of fluid flow

Fundamental properties of the motion of a fluid such as air or water are its density, velocity and pressure. For example, consider a weather map. On it are plotted isobars — curves of constant pressure at the Earth's surface — which are also intimately related to the strength and direction of the wind — its velocity. These properties are of prime importance in weather forecasting.

In our application to fluid dynamics these fields also depend upon time t.

The crucial aspect of the density, velocity and pressure fields is that at each and every point in space, denoted generically by P, a value for the field exists: a vector value for the velocity field, and a scalar value for the pressure or density. These are examples of what we call vector functions of position, denote velocity by $\mathbf{v}(P)$ for example, and scalar functions of position, denote density and pressure by $\rho(P)$ and $p(P)$ respectively.

Objectives:

- to recall how to describe and manipulate scalar and vector fields in a Cartesian[1] coordinate system;
- to introduce the fluid fields of pressure, density and velocity;
- to recall level curves, level surfaces and field lines and see their application to fluid flow.

Reading 1.B → Study Section 9.4 in Kreyszig [Kre06, pp. 384–389]

Activity 1.C → Do a selection of problems from Problem Set 9.4 [Kre06, pp. 389]. Send in to the lecturer for feedback solutions for Problems 7 and 19.

We have to understand whatever we mathematically analyse. Thus it is important to be able to visualise the structures represented by any given algebraic expression, or by any numerical approximation. Drawing *level curves*, or contour maps, are one means of visualising scalar functions. For example, the isobars on a weather map are the level curves of atmospheric pressure.

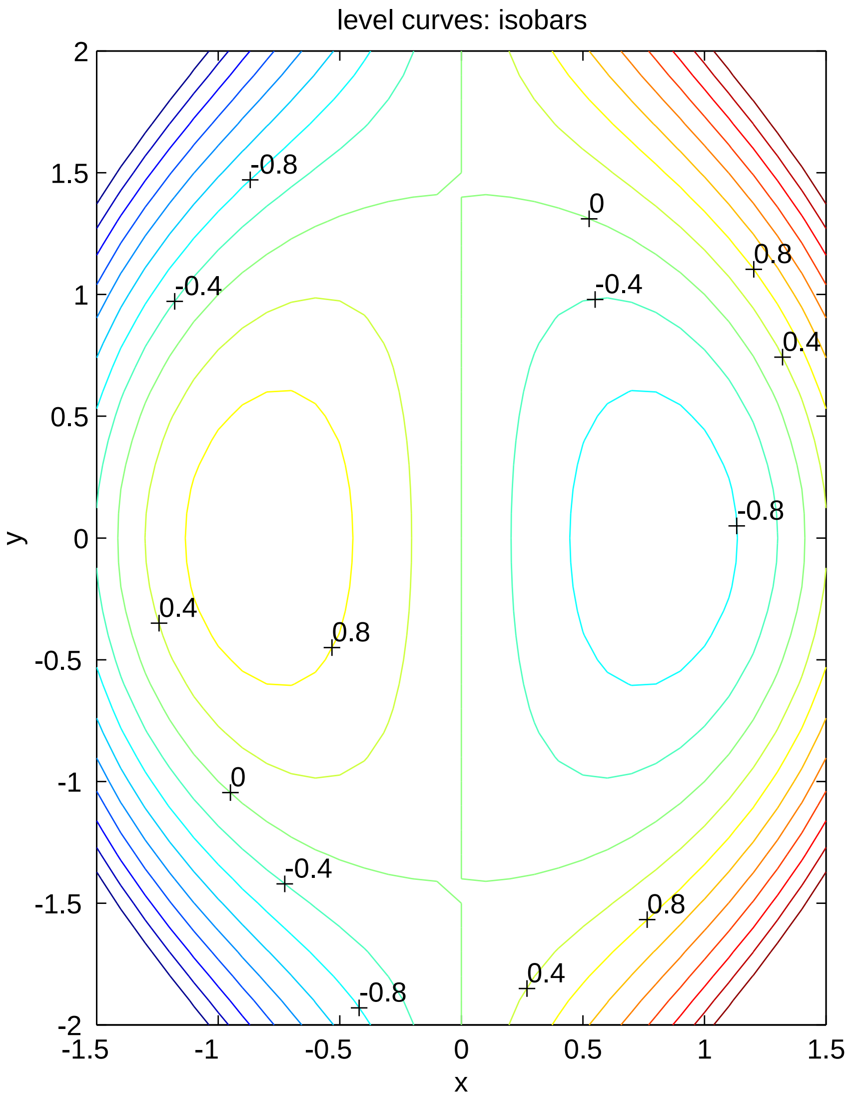
Example 1.1: level curves Consider the (artificial) two-dimensional pressure field given algebraically by

$$p = x^3 + xy^2 - 2x\,.$$

The level curves, or contours or isobars, of p are plotted in the following figure. These days such plots are easy to do on computers using software tools such as MATLAB as indicated by the commands listed beside the picture.

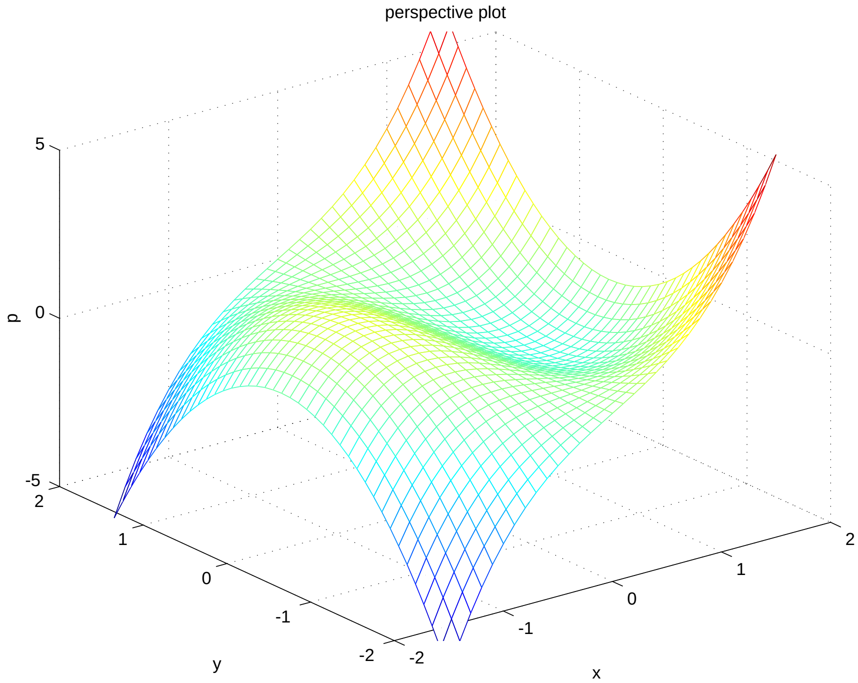
Develop the habit of visualising algebra by using the computer to draw pictures.

[1] René Descartes (1596–1650) was a celebrated French philosopher and mathematician. Descartes greatest work in mathematics was the development of analytical geometry — the association of points in space with numerical coordinates and algebraic equations. This close linking together of algebra and geometry is the foundation of huge areas of mathematics, and vector calculus in particular.




```
[x,y]=meshgrid(-1.5:0.1:1.5, ...
-2:0.1:2);
p=x.^3-2*x+x.*y.^2;
clabel(contour(x,y,p));
```

A perspective view of the same pressure field is shown below.




```
mesh(x,y,z)
```

This shows the same features but using a different style of display. The level curves may be viewed as the intersection of this surface with planes of constant pressure p as in the animation shown by `isobar2.m` (available on electronic sources).

Algebraically, a level curve is found by setting p to be a constant in the formula above to obtain an equation for a set of points in space. For example, the level curve $p = 0$ is found as follows.

- It is the solution of the equation $0 = x^3 + xy^2 - 2x$.
- Factoring x leads to the form $0 = x(x^2 + y^2 - 2)$.
- Thus $p = 0$ if either $x = 0$ or $x^2 + y^2 = 2$.
- As seen in the contour plot, the level curve $p = 0$ is the union of the straight, vertical line $x = 0$, and the circle of radius $\sqrt{2}$ centred at the origin.

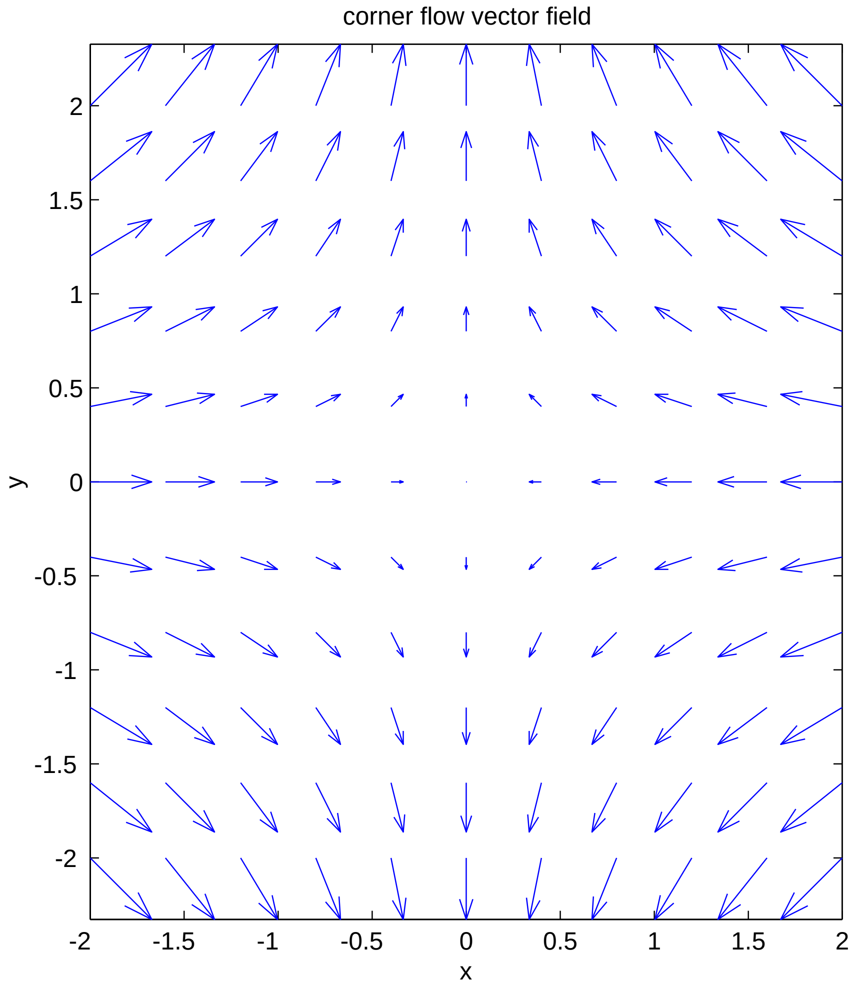
Other level curves are more difficult to visualise from the algebra. In general, the level curve $p = C$ for some constant C is obtained similarly.

- It is the solution of $C = x^3 + xy^2 - 2x$.
- One rearrangement leads to the expression $y = \pm\sqrt{\frac{C+2x-x^3}{x}}$.

This family of curves has no well recognised name, but for any constant C we may compute and plot the level curve. See in the given contour plot that there is a localised region of low pressure just to the right of the origin, and a localised region of high pressure just to the left.

Recall that nested closed level curves, such as those around $(\pm\sqrt{2/3}, 0)$, indicate a local maximum or minimum in the field. Conversely, intersecting level curves, such as those at $(0, \pm\sqrt{2})$, indicate a saddle point. See these features in the perspective plot.

Example 1.2: vector fields and streamlines Consider a fluid's velocity field $\mathbf{v} = -x\mathbf{i} + y\mathbf{j}$ as displayed below.




```
[x,y]=meshgrid(-2:0.4:2);
u=-x;
v=y;
quiver(x,y,u,v)
```

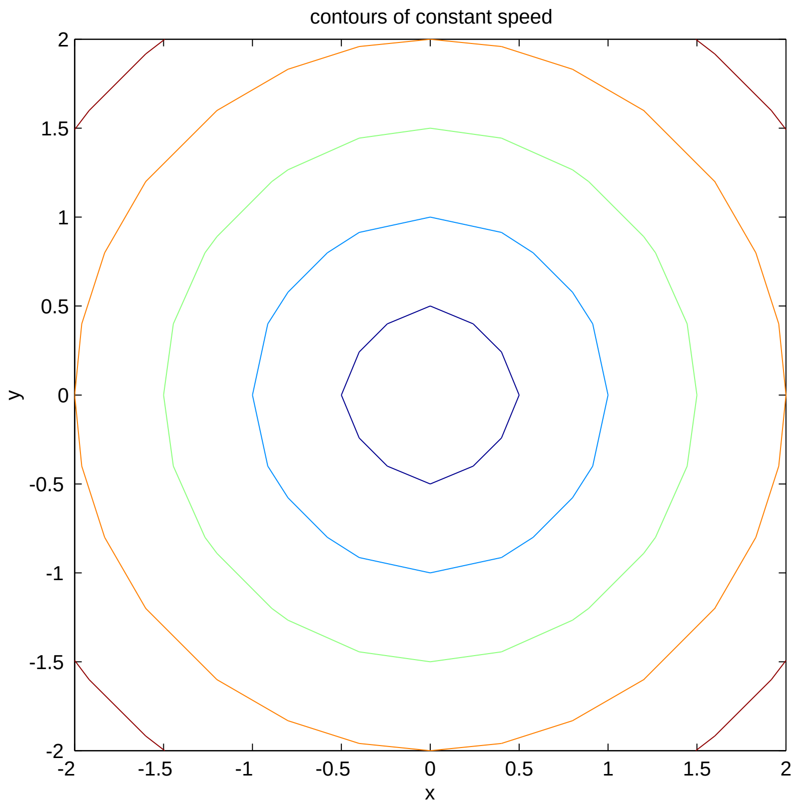
Restricted to the first quadrant, this may represent the wind blowing in off the sea (water surface at the x-axis) and then having to turn up a vertical cliff (the cliff is along the y-axis). The crucial point here is that *every* point in the plane has a vector associated with it. In this application the vector gives the direction and speed of the fluid at that point. For example:

- at the point $(1, 1)$ the wind is blowing with velocity $\mathbf{v} = -\mathbf{i}+\mathbf{j}$, that is, at speed $\sqrt{2}$ in a direction to the left and up at $45°$ to the horizontal;

- at the point $(2, 0)$ the wind blows with velocity $\mathbf{v} = -2\mathbf{i}$, that is, directly to the left at speed 2.

Of *little* practical significance are the curves on which the wind has the same speed, and the curves on which the wind has the same direction. Yet they can be used to visualise the vector field.

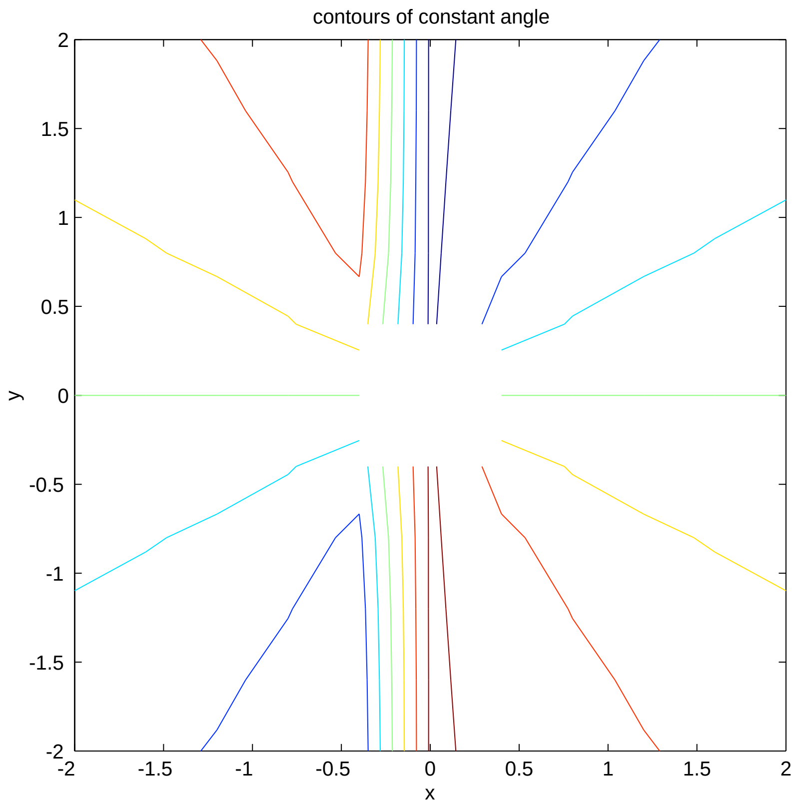
- Curves where the wind has the same speed are curves for which $|\mathbf{v}|$ is constant, say C. As we usually do, and introducing this notation is probably the most important aspect of this exercise, let the Cartesian components of the velocity field be denoted by u and v (and w in three dimensions), that is, $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$. Then in this example, $u = -x$ and $v = y$ so that $|\mathbf{v}| = \sqrt{u^2 + v^2} = C$ reduces to simply $\sqrt{x^2 + y^2} = C$ which we recognise as circles of radius C centred upon the origin as shown below




```
contour...
(x,y,sqrt(u.^2+v.^2))
```

- Curves on which the wind is in the same direction are ones for which $\mathbf{v}$ is at the same slope to the horizontal, say at a slope $C = \tan\theta$ for some angle θ. Since u is the horizontal component and v the vertical component, trigonometry asserts that $v/u = \tan\theta = C$. Hence here $y/(-x) = C$ which is rearranged to be $y = -Cx$. That is, curves of constant direction are the straight lines through the origin. MATLAB makes a little mess drawing these but the following code draws roughly the correct picture:

atan is the inverse tangent function $\tan^{-1}$.




```
contour...
(x,y,atan(v./u))
```

Of importance are the *streamlines* of the fluid flow, or *field lines* as they are known in general. Why are streamlines significant? Because the streamlines tell us where the fluid particles move over time. Of course such useful information is not obtained cheaply; in general we have to solve a differential equation.

Now the fluid velocity points in the direction $\mathbf{v}$, this being the direction of motion of the particles of the fluid. Thus if the path of any one fluid particle is described by a function $y = y(x)$, then the vector field, the fluid's velocity, must point *along* the curve. That is, at every point of the curve, the slope of the curve, dy/dx, must be the same as the slope of the velocity vector, namely v/u. Thus to find *streamlines* solve the first-order ordinary differential equation (ODE)

$$\frac{dy}{dx} = \frac{v}{u}. \tag{1.1}$$

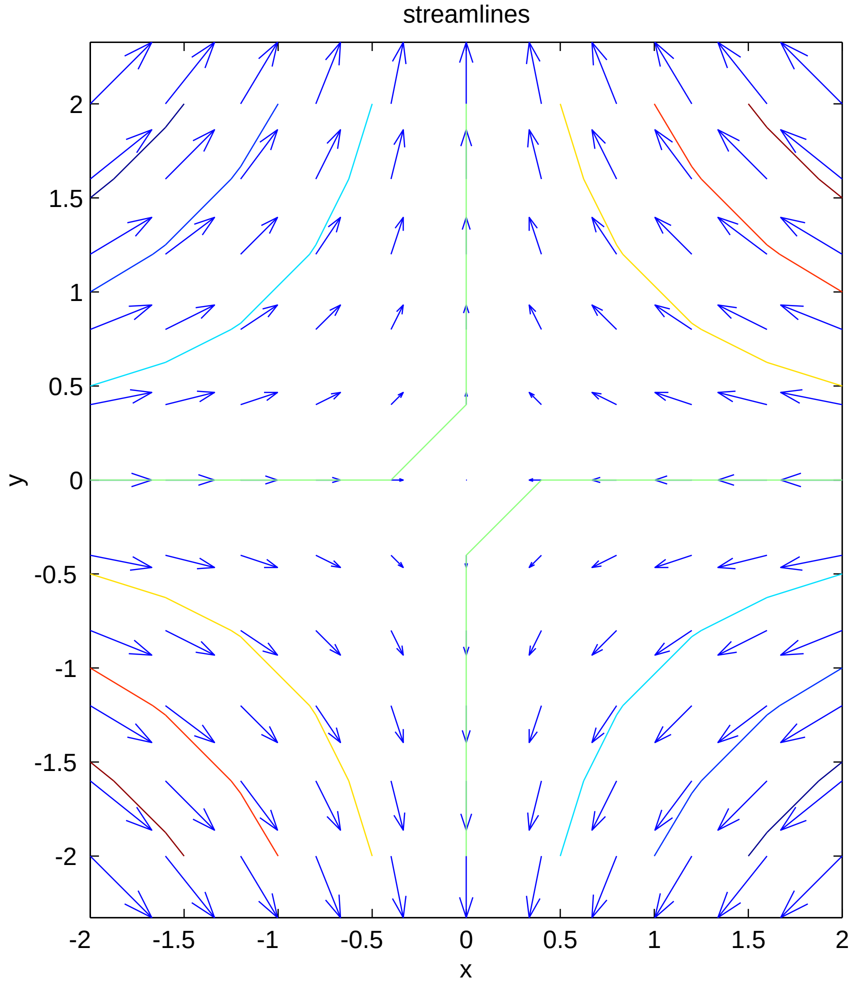
Here for example:

- $u = -x$ and $v = y$ so that we solve the ODE $\frac{dy}{dx} = -\frac{y}{x}$;
- this is a separable ODE as it is $\frac{dy}{y} = -\frac{dx}{x}$;

- write in integral signs as $\int \frac{dy}{y} = -\int \frac{dx}{x}$;
- then integrate to $\ln y = -\ln x + C'$
- and taking the exponential of both sides shows $y = C/x$ (where $C = \exp(C')$).

Note: throughout we will use $\ln$ to denote the natural logarithm (to base e) as it is *the* important logarithm. If ever needed, the logarithm to the base 10 will be denoted by $\log_{10}$.

Recognise that this formula describes a family of hyperbola with x-axis and y-axis as horizontal and vertical asymptotes respectively. See below for a plot of these streamlines obtained as the contours of the function xy since the obtained algebraic solution may be rewritten as $xy = C$:




```
quiver(x,y,u,v)
hold on
contour(x,y,x.*y)
hold off
```

The streamlines show that fluid particles approaching from the right and above the x-axis are swept up in front of the vertical "wall" of the y-axis. For example, the fluid particles passing through the point $(2, 1)$ travel along the hyperbola $xy = 2$ ($C = 2$ here because we know $x = 2$ and $y = 1$ are on the hyperbola and xy is then 2).

Note that different values of the integration constant C give different streamlines corresponding to the different paths that the fluid particles take. This is true in general. The integration constant (here C) obtained in solving the differential equation (1.1) parameterises all the possible streamlines in the flow.

As an important application, suppose that this velocity field, $\mathbf{v} = -x\mathbf{i} + y\mathbf{j}$, described the wind near the Earth's surface at about the time the Chernobyl nuclear reactor exploded and released its poisonous cloud of radioactive material. If, say, the reactor was located at $(2, 1)$, then the cloud of contaminated air, carried by this wind, would follow the streamline $xy = 2$ because that is the path taken by the air passing by the point $(2, 1)$. Thus streamlines help us predict where such pollutant moves.

Finally we note without derivation that in three-dimensional Cartesian coordinates, streamlines are obtained by solving the coupled differential equations obtained by any pair of equalities in

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

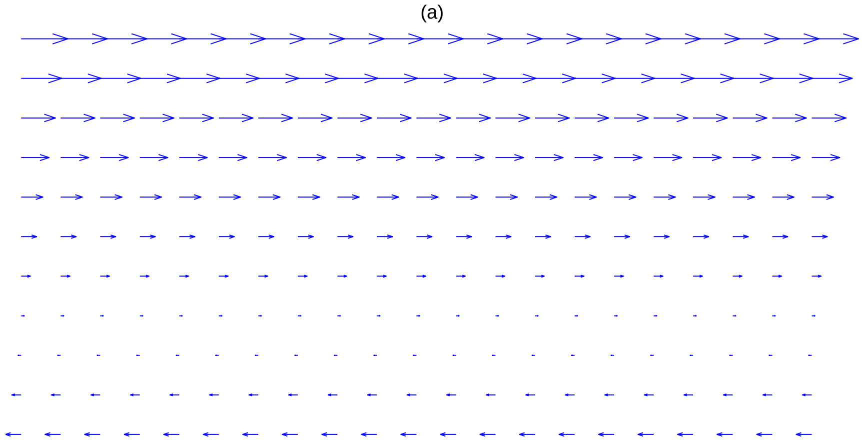
Activity 1.D → Do the following exercises. Send in to the lecturer for feedback solutions for Problem 17 in Exercise 1.3, and Exercises 1.4(b) and (c) below.

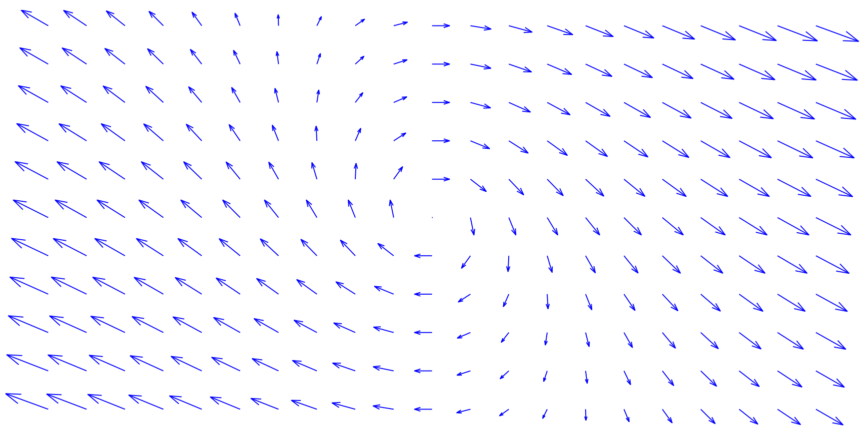
1.1.1 Field exercises

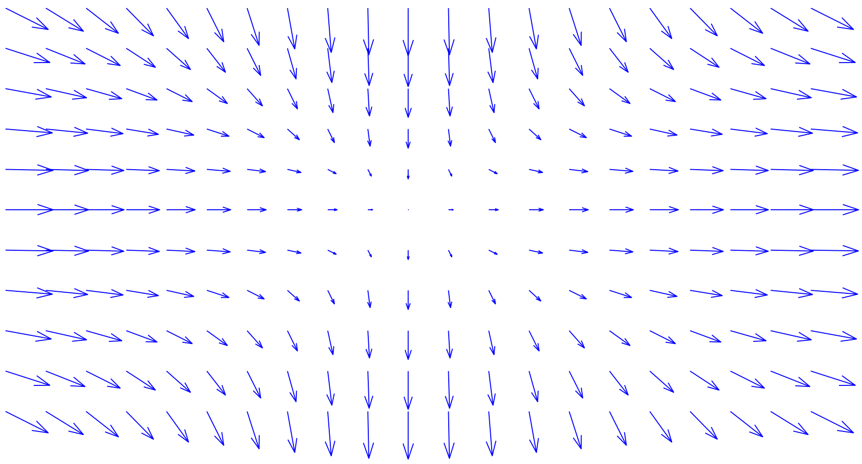
Ex. 1.3: Find algebraic expressions and plot field lines for the vector fields (streamlines if it is a fluid flow) in Problems 15–20 from Problem Set 9.4 [Kre06, pp. 389]. For each of these problems, assume that $\mathbf{v}$ describes a fluid velocity field and determine where would a pollutant be carried if it was released at the location $(2, 1)$?

Ex. 1.4: Sketch representative streamlines for the following vector fields:

(a)







Note that in this last exercise no coordinate axes or labels are drawn. This is because the important concepts developed in this course depend only the spatial structure of the vector and scalar fields. Algebraic manipulation of descriptions based upon any coordinate system is only *one* method of mathematics, in many cases graphical understanding is more fundamental.

1.2 Material derivatives and the gradient

A fluid moves. Thus fluid particles "see" the world from different places at different times. Hence we need to consider the rate of change in quantities such as density and pressure as felt by a fluid particle in its movement. For example, in the velocity field $\mathbf{v} = -x\mathbf{i} + y\mathbf{j}$ at the point $(1, 2)$ the fluid is moving with velocity $-\mathbf{i} + 2\mathbf{j}$. Hence in a pressure field such as $p = x^3 - 2x + xy^2$, as the fluid particle moves it will feel a varying pressure (in this case increasing at the rate 3) and such variations affect its motion. Thus we often need to differentiate in the direction of the fluid flow. Such a *material derivative* is the operation of fundamental importance in continuum mechanics.

Because of the chain rule, finding derivatives of a scalar function in any given direction introduces the *gradient* of the scalar function. The gradient turns out to be immensely useful. The gradient is the fundamental differential operator in vector calculus. We must understand its properties.

Objectives:

$\boldsymbol{\nabla}$ is Greek symbol "nabla"

- to develop the properties of the gradient, $\boldsymbol{\nabla}$ or "grad", namely that $\boldsymbol{\nabla} f$ points in the direction of maximum increase of f and is of magnitude equal to the rate of change of f in that direction;
- to use the gradient in Cartesian coordinates, namely
$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\,,$$
for directional derivatives, surface normals and velocity potentials;
- to introduce the material derivative of fluid flow.

Reading 1.E → Study Section 9.7 in Kreyszig [Kre06, pp. 403–410].

Activity 1.F → Do a selection of problems from Problem Set 9.7 [Kre06, pp. 409–410]. Send in to the lecturer for feedback solutions for Problems 30 and 34.

Also see the material mirrored at http://www.math.montana.edu/frankw/ccp/multiworld/twothree/gradient/learn.htm.

1.2.1 The material derivative

Suppose we are interested in what a specific particle or drop of fluid experiences as it moves through space. The fluid particle moves along some path described by $\mathbf{r}(t) = x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$, a description in which its position is parameterised by time t. Any scalar field that it moves through, such as pressure, density or temperature, is given by some function of position and, for generality, time; say $f(x, y, z, t)$. Then the given fluid particle experiences, or "sees", only $f = f(x(t), y(t), z(t), t)$. Differentiating this with respect to time then gives the rate of change of f as observed by the fluid particle, denoted by Df/Dt and called the *material derivative*. But the chain rule tells us that

$$\begin{aligned}\frac{Df}{Dt} &= \frac{d}{dt} f(x(t), y(t), z(t), t)\\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\\ &= \frac{\partial f}{\partial t} + \frac{d\mathbf{r}}{dt}\cdot(\boldsymbol{\nabla} f)\,.\end{aligned}$$

However, for a fluid particle, $d\mathbf{r}/dt$ is just the fluid velocity $\mathbf{v}$. Hence the material derivative of a scalar function f is

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f\,. \tag{1.2}$$

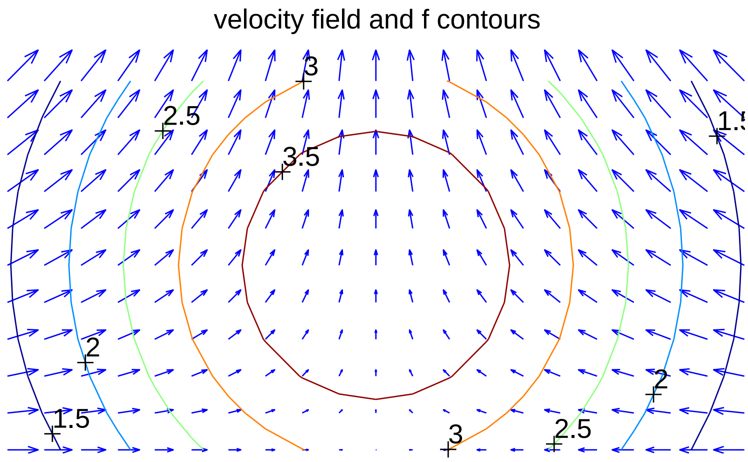


Figure 1.1: Velocity field and contours of function f.

There is no time dependence in this example.

Example 1.5: perceived pressure The pressure field $p = x^3 - 2x + xy^2$ has gradient $\boldsymbol{\nabla} p = (3x^2 - 2 + y^2)\mathbf{i} + 2xy\mathbf{j}$. Hence the fluid particle instantaneously at $(1, 2)$ in the velocity field $\mathbf{v} = -x\mathbf{i} + y\mathbf{j}$ experiences a pressure which is changing at a rate

$$\frac{Dp}{Dt} = \mathbf{v} \cdot \boldsymbol{\nabla} p = (-\mathbf{i} + 2\mathbf{j}) \cdot (5\mathbf{i} + 4\mathbf{j}) = 3\,.$$

This perceived variation in time of the pressure is solely due to the fluid's movement through a spatially varying pressure field. With the formula at hand we may of course compute the rate of change for fluid particles at an arbitrary point, namely

$$\frac{Dp}{Dt} = \mathbf{v} \cdot \boldsymbol{\nabla} p = (-x\mathbf{i} + y\mathbf{j}) \cdot [(3x^2 - 2 + y^2)\mathbf{i} + 2xy\mathbf{j}] = -3x^3 + 2x + xy^2\,.$$

Example 1.6: graphical Consider the scalar function $f(x, y)$ whose labelled contours are displayed in Figure 1.1, and the vector velocity field $\mathbf{v}$ superimposed. With no time dependence, the material derivative Df/Dt measures the observed changes in f when one moves in the direction of the velocity field. Thus, as shown in Figure 1.2,

- Df/Dt is positive wherever the velocity arrows point up across contours of f;
- Df/Dt is negative wherever the velocity arrows point downwards across contours of f;
- and Df/Dt is approximately zero wherever the velocity is near zero (middle bottom), f is not varying (the top of the hill), or the velocity arrows are parallel to the contours of f.

Fluid particles satisfy a form of Newton's law of motion: an applied force generates an acceleration of the particle. We will see this arise in Section 1.6 when we consider conservation of momentum. But the acceleration is that of the particles of fluid and hence we need to determine what

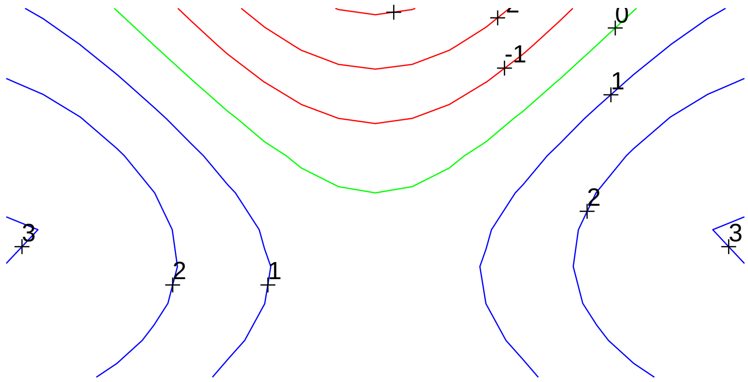


Figure 1.2: Material derivative of function f.

is the change of fluid velocity as *seen* by a fluid particle. By the same reasoning as above, this is also given by the material derivative, namely

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\,. \tag{1.3}$$

You should wonder precisely what $\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$ actually means. The convention is that implicit brackets exist so that in Cartesian coordinates it is computed as

The right-hand side is different in curvilinear coordinates such as spherical or cylindrical coordinates.

$$\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = (\mathbf{v}\cdot\boldsymbol{\nabla})u\mathbf{i} + (\mathbf{v}\cdot\boldsymbol{\nabla})v\mathbf{j} + (\mathbf{v}\cdot\boldsymbol{\nabla})w\mathbf{k}\,,$$

where $(\mathbf{v}\cdot\boldsymbol{\nabla}) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$. The reason is that the acceleration in the x-direction, that of $\mathbf{i}$, is simply the material derivative of the velocity u in the x-direction. Similarly for the other two directions, leading to the expression given above which is also frequently written in a component form as

$$\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = \begin{pmatrix} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{pmatrix}.$$

Example 1.7: In the velocity field $\mathbf{v} = -x\mathbf{i} + y\mathbf{j}$,

$$\mathbf{v}\cdot\boldsymbol{\nabla} = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,,$$

and so the acceleration of all the fluid particles is

$$\frac{D\mathbf{v}}{Dt} = \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = \left(-x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\mathbf{i} + \left(-x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)\mathbf{j} = x\mathbf{i} + y\mathbf{j}\,.$$

For example, the fluid particle at $(1,1)$ is moving up and to the left with velocity $\mathbf{v} = -\mathbf{i} + \mathbf{j}$. But it also lies on a streamline which curves up and to the right as you see in the graph of the streamlines. Thus its acceleration must be up and to the right; the above formula tells us that the acceleration of the fluid particle at $(1,1)$ is precisely $\mathbf{i} + \mathbf{j}$ which agrees with this qualitative deduction from the figure.

1.2.2 The velocity potential

As described in Kreyszig [Kre06, pp. 407–409], some vector fields may be written as the gradient of a potential. For example, the fluid velocity field

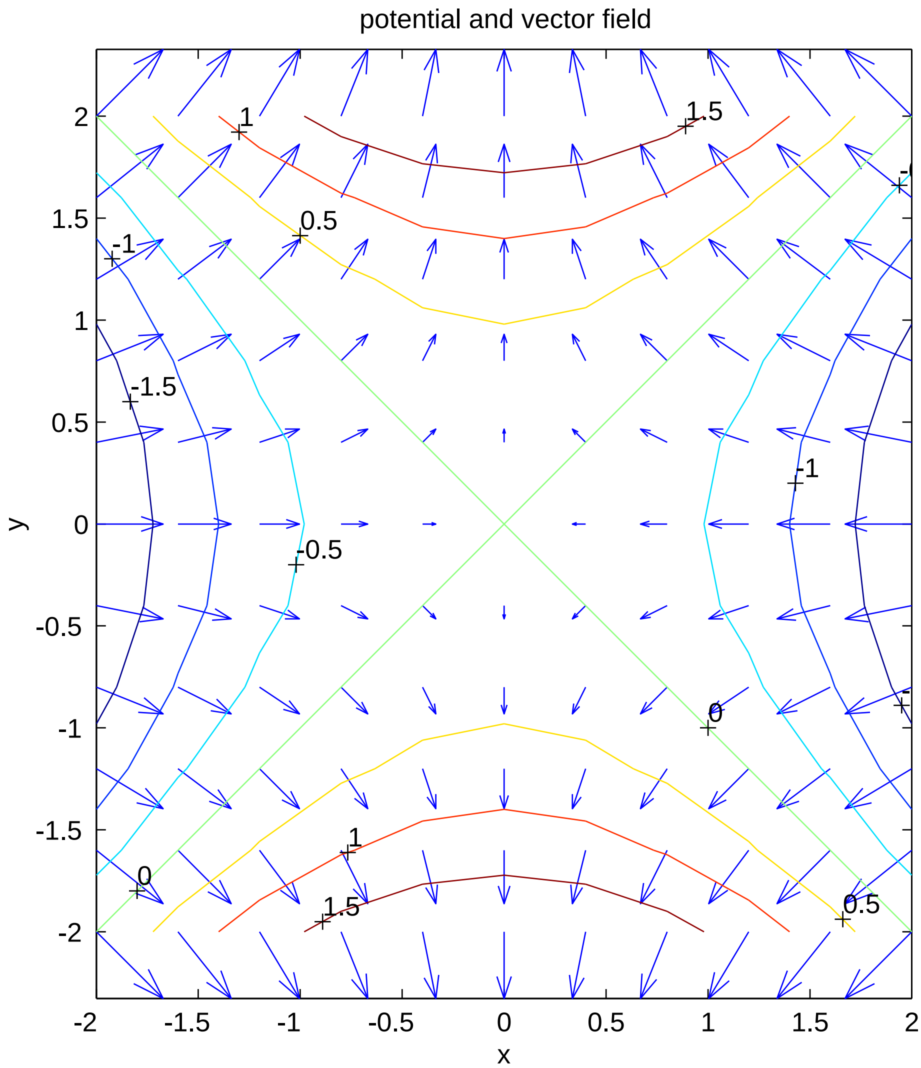
$$\mathbf{v} = -x\mathbf{i} + y\mathbf{j} = \boldsymbol{\nabla}\left(-\frac{1}{2}x^2 + \frac{1}{2}y^2\right)$$

φ is Greek letter "phi"

is the gradient of the *velocity potential*, $\varphi = -\frac{1}{2}x^2 + \frac{1}{2}y^2$. See in the following picture how

- the velocity field (the gradient) is normal to the contours of this velocity potential;
- the speed of the fluid (the arrow length) is proportional to the rate of change of the potential (inversely proportional to the spacing between the contours);
- and points to higher values of φ.

The `axis('equal')` command is essential to ensure angles appear correctly in such a plot.




```
[x,y]=meshgrid(-2:0.4:2);
```

```
quiver(x,y,-x,y,'r')
axis('equal')
hold on
phi=(-x.^2+y.^2)/2;
clabel(contour(x,y,phi))
hold off
```

Indeed, in an important class of fluid flows, called irrotational as we shall see in Section 1.5, the velocity field may be *always* written as the gradient of a velocity potential. The advantage of this approach is that complete vector velocity field is given by a single scalar function.

Example 1.8: When the (flow) potential function is given the corresponding gradient (velocity) field is easy to obtain by direct differentiation of the potential and finding its gradient. However the inverse problem of finding the potential for a given vector field is more complicated. Firstly, not all vector fields have the potential, see Section 1.5, and, secondly, finding the potential requires solving a system of partial differential equations. Suppose that the given velocity field is $\mathbf{v} = (u, v) = (-x, y)$ and it is known that it has the potential φ. How to find it?

By definition,

$$\boldsymbol{\nabla}\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) = (u, v) = (-x, y), \text{ i.e.}$$

$$\frac{\partial\varphi}{\partial x} = -x\,, \quad \frac{\partial\varphi}{\partial y} = y\,.$$

Integrating the first equation with respect to x we obtain

$$\varphi = -\frac{x^2}{2} + f(y)\,.$$

Note the appearance of an arbitrary function $f(y)$ in this integration. This is due to the fact that the equation we just integrated was a *partial* differential equation in x. Integrating the second equation in y we obtain

$$\varphi = \frac{y^2}{2} + g(x)\,.$$

Since both answers should give rise to the same function we must enforce that

$$\varphi = -\frac{1}{2}x^2 + f(y) = \frac{1}{2}y^2 + g(x)\,.$$

By inspection we conclude that this can only be satisfied if

$$f(y) = \frac{1}{2}y^2 + C \text{ and } g(x) = -\frac{1}{2}x^2 + C$$

so that

$$\varphi = -\frac{1}{2}x^2 + \frac{1}{2}y^2 + C\,.$$

The numerical value of the integration constant C is not important because only the gradient of potential, not the potential itself, has physical importance and the presence or absence of the constant does not affect the gradient.

Activity 1.G $\rightarrow$ Do the following exercises. Send in to the lecturer for feedback solutions for Exercises 1.9, 1.11, 1.12, 1.13(b), 1.14(b) and 1.15(b).

1.2.3 Graded exercises

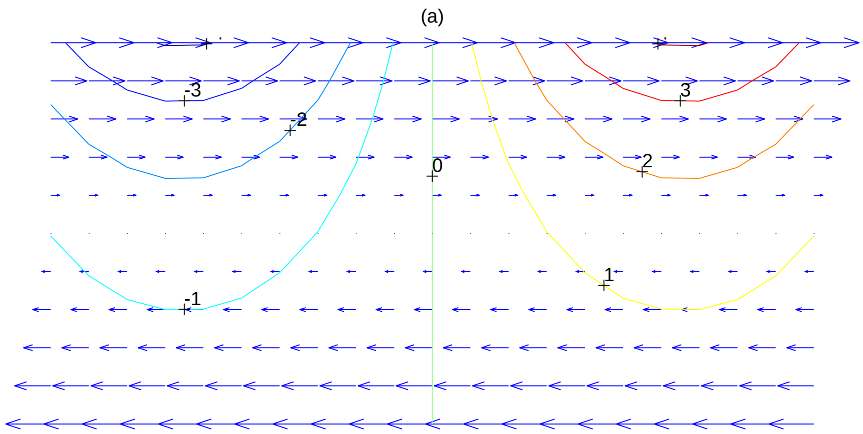
Ex. 1.9: Show that the direction of the gradient of a scalar field is orthogonal to the level curves of that field.

Ex. 1.10: Review and list the conditions for existence of the potential of a vector field.

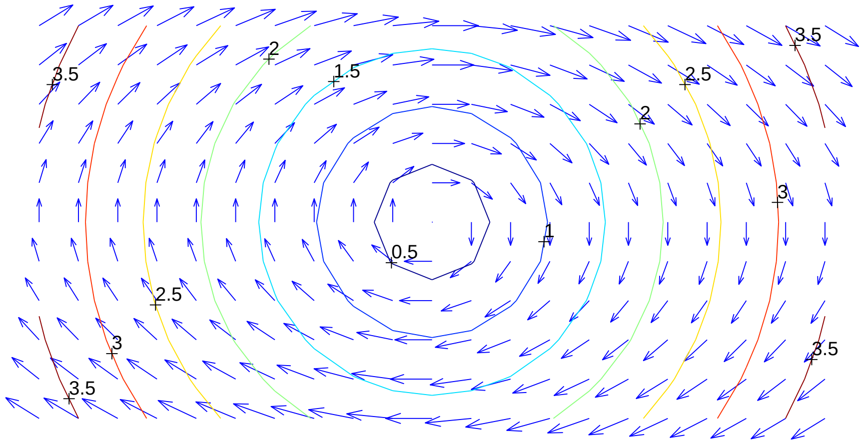
Ex. 1.11: Can flow potential be introduced for three-dimensional flows? If so, how is it related to the velocity components?

Ex. 1.12: Derive an expression for the velocity potential φ for a uniform flow $\mathbf{v} = (u, v) = (V\cos\alpha, V\sin\alpha)$, where V and α are constants. What is the equation describing equi-potential lines? Sketch a few lines of constant velocity potential.

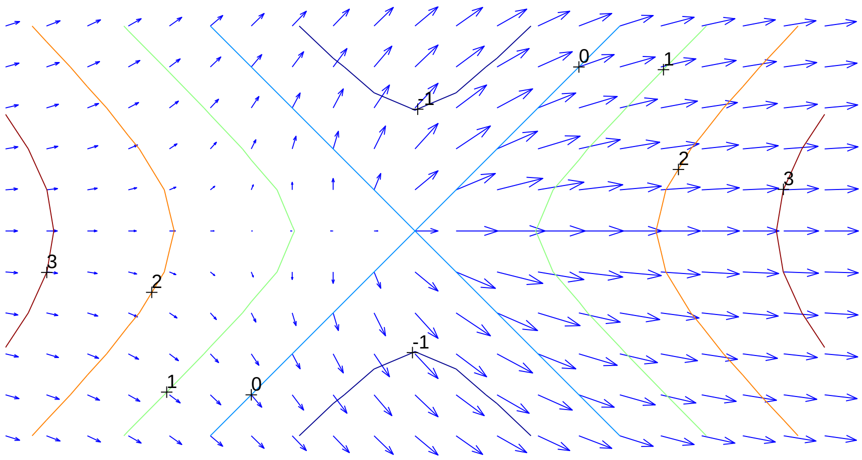
Ex. 1.13: Given the plotted level curves of a scalar function f and the plotted velocity field $\mathbf{v}$, sketch the regions where the material derivative Df/Dt is positive, negative and approximately zero respectively (assume $\partial f/\partial t = 0$).



(b)

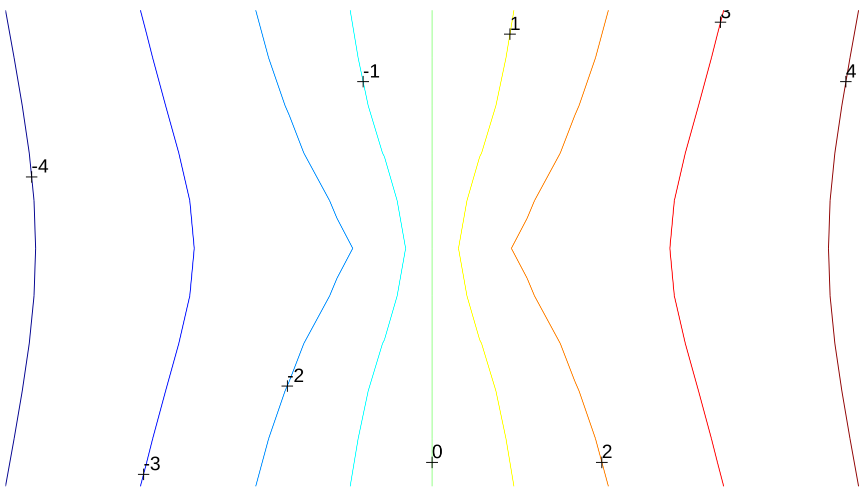


(c)

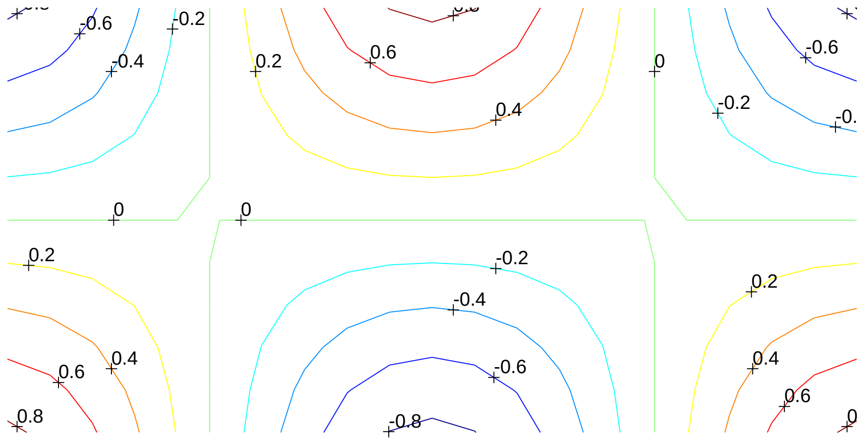


Ex. 1.14: For each of the scalar functions whose level curves are plotted below, sketch in the vector field of its gradient — endeavour to get the directions and relative magnitudes correct.

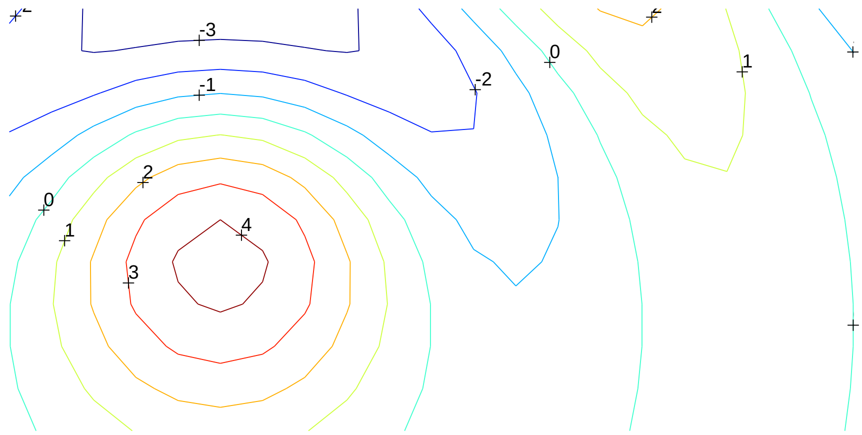
(a)



(b)

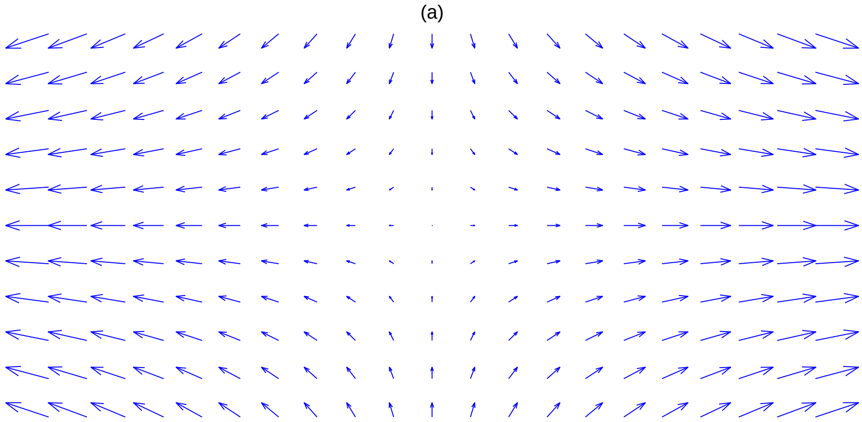


(c)

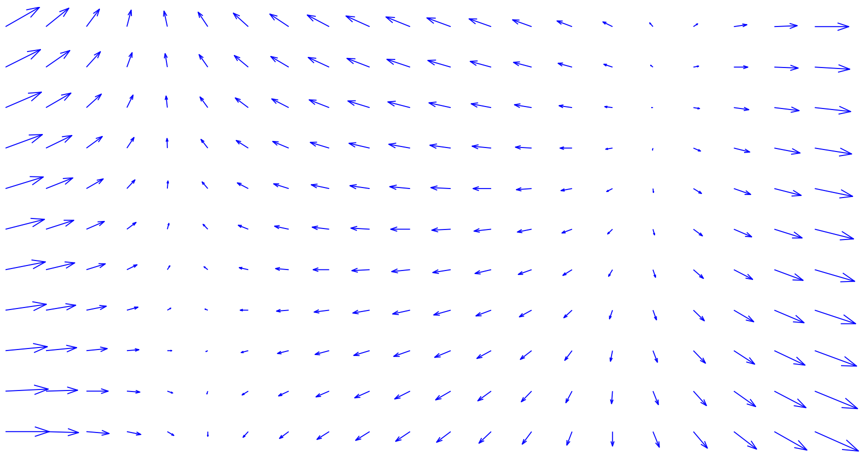


Ex. 1.15: Given that the following vector fields are the gradient of a scalar potential, sketch level curves of the potential and indicate where the potential is a local maximum or minimum.

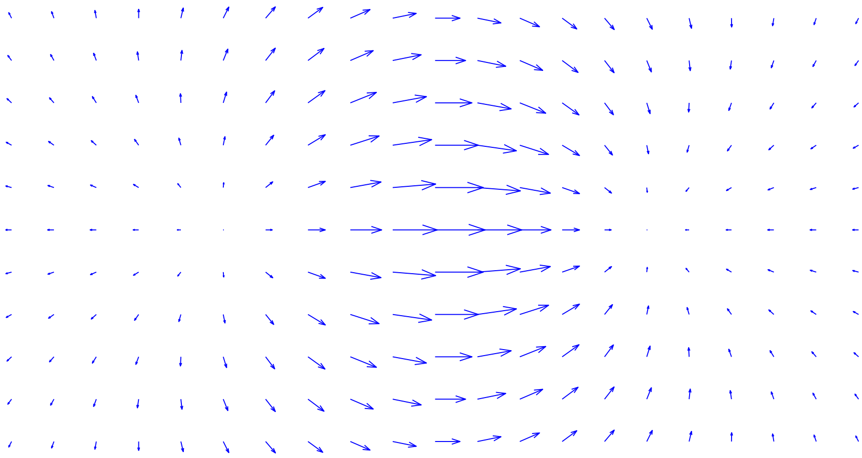
(a)



(b)



(c)



1.3 Divergence does not conserve volume

Q: How can you tell that Harvard was laid out by a mathematician?
A: The div school [divinity school] is right next to the grad school...

$\nabla\cdot$ is pronounced "div".

The second fundamental differentiation operator in vector calculus is that of the *divergence*. Computed in Cartesian coordinates by the formula, that if $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, then

$$\operatorname{div}\mathbf{v} = \nabla\cdot\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{1.4}$$

it may be simply applied to determine, for example, that

$$\nabla\cdot(-x\mathbf{i} + y\mathbf{j}) = \frac{\partial(-x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(0)}{\partial z} = -1 + 1 = 0\,.$$

Here the divergence just happens to be everywhere zero. In general the divergence acts upon a vector function and the result is a scalar function of position.

Objectives:

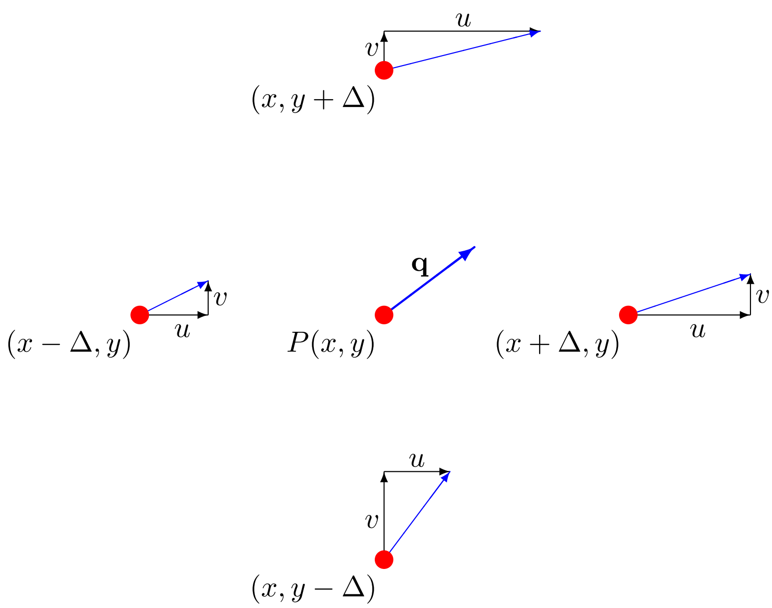
- to refresh and use your knowledge of the divergence;
- to understand the significance of the divergence in the conservation of mass and volume;
- to derive the continuity equations of compressible flow and of incompressible flow.

Reading 1.H → Study Section 9.8 in Kreyszig [Kre06, pp. 410–413]. Especially study EXAMPLE 2 which derives the continuity equation.

1.3.1 Why is it called the divergence?

Consider the flow of an incompressible fluid — one, like water, that can neither expand or compress. If some of the fluid is moving away from any given point, then other fluid must move in to fill what otherwise would be a vacuum. Thus on average there must be just as much fluid moving in towards a point, as moving away from it. The divergence measures this balance precisely.

Very loosely speaking, the divergence of a vector field at a point P measures the amount that the vector field "points away" from P, hence its relevance to fluid flow and also hence its name of "divergence". See this very crudely in two dimensions via the following argument. Choose any point $P(x, y)$ and consider a velocity field $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$ at four neighbouring points as shown below.



What do we mean by a vector field "pointing away"? It means the average around neighbouring points of the component of $\mathbf{v}$ directed away from the central point P. Here:

- to the right at $(x + \Delta, y)$ the component is $u(x + \Delta, y)$ as the v component is directed neither towards nor away from P;

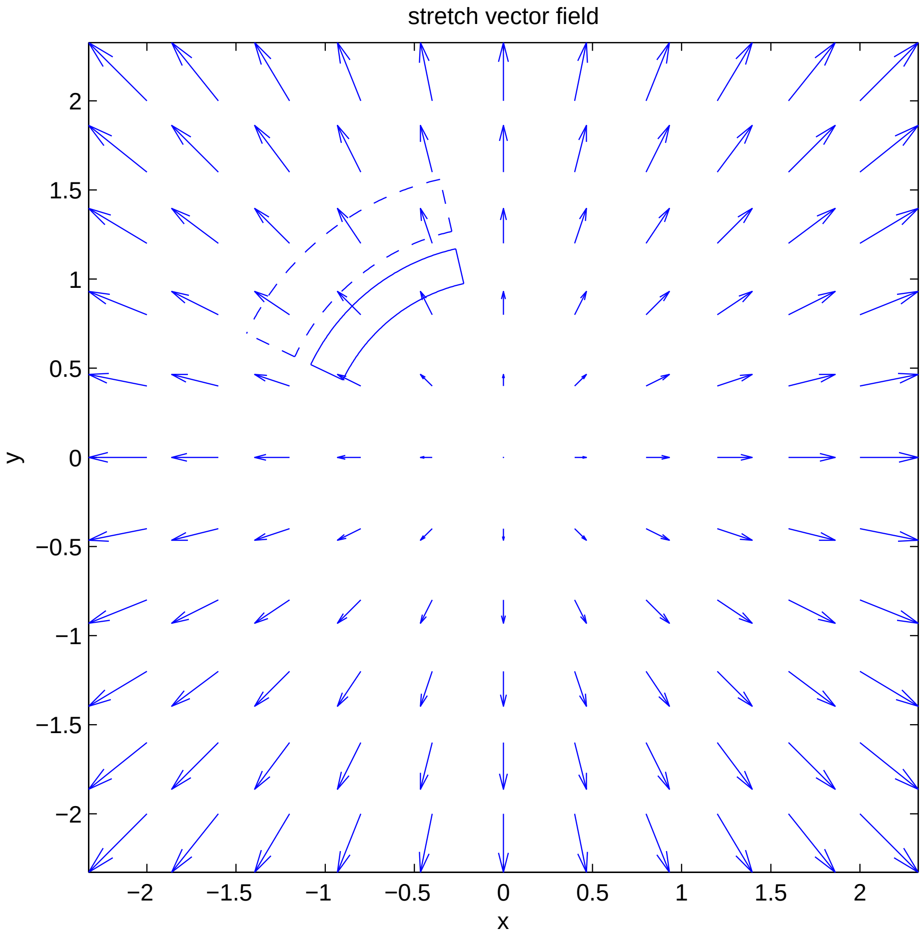
- above at $(x, y+\Delta)$ the component is $v(x, y+\Delta)$ as here it is the u component that is neither towards nor away from P;
- to the left at $(x-\Delta, y)$ the component is $-u(x-\Delta, y)$ (minus because positive u points towards P);
- below at $(x, y-\Delta)$ the component is $-v(x, y-\Delta)$ (minus because positive v points towards P).

Thus the average of these "pointing away" components is

$$\begin{aligned}
& \frac{1}{4}\left[u(x+\Delta, y)+v(x, y+\Delta)-u(x-\Delta, y)-v(x, y-\Delta)\right] \\
= & \frac{\Delta}{2}\left[\frac{u(x+\Delta, y)-u(x-\Delta x, y)}{2\Delta}+\frac{v(x, y+\Delta)-v(x, y-\Delta)}{2\Delta}\right] \\
\approx & \frac{\Delta}{2}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right] \quad \text{by a limit definition of the derivatives} \\
& = \frac{\Delta}{2}\boldsymbol{\nabla}\cdot\mathbf{v} \quad \text{by Cartesian formula for divergence.}
\end{aligned}$$

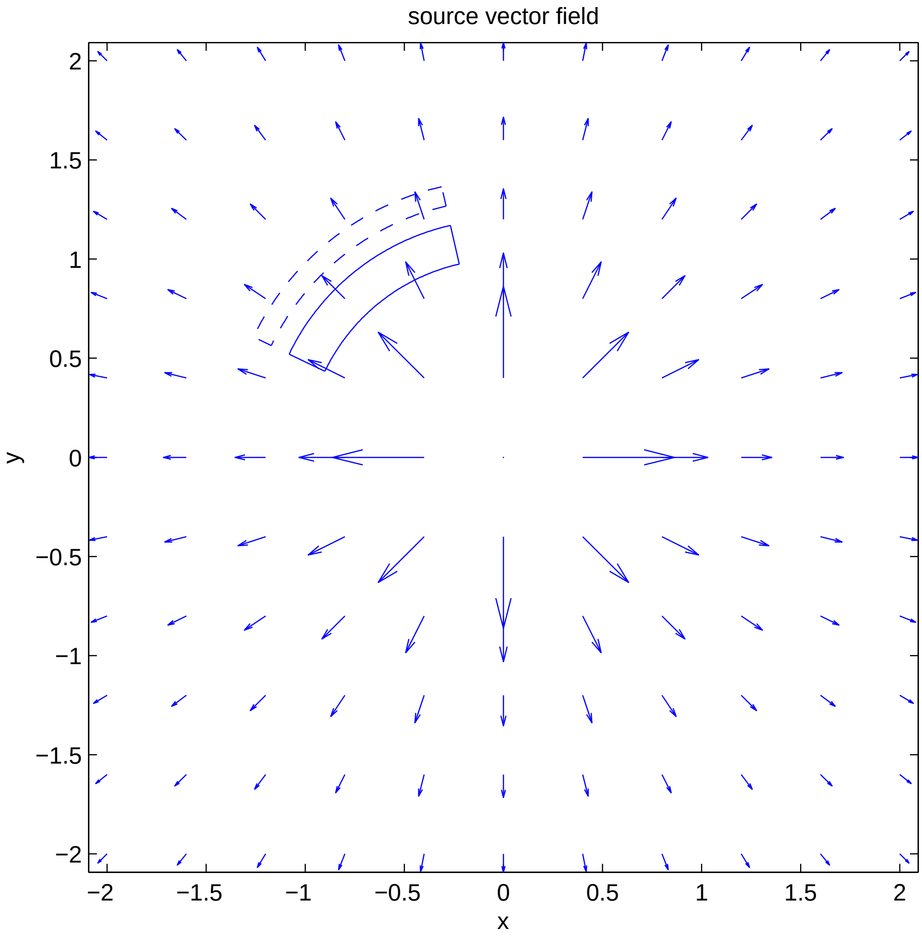
This demonstrates that the divergence of a vector field at any point is proportional to the average of how much the vector field "points away" from the point. In the flow of an incompressible fluid, there must be just as much fluid moving in as moving out — the divergence of the fluid's velocity field thus has to be zero. In essence this is the same property of the divergence as discussed in EXAMPLE 2 of Kreyszig [Kre06, pp. 412–413].

For example, contrast the vector field $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ shown below




```
[x,y]=meshgrid(-2:0.4:2);
quiver(x,y,x,y)
```

for which the divergence is everywhere $\nabla\cdot\mathbf{v} = 2$, with the vector field $\mathbf{v} = (x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$ shown here




```
[x,y]=meshgrid(-2:0.4:2);
u=x./(x.^2+y.^2);
v=y./(x.^2+y.^2);
quiver(x,y,u,v)
```

for which the divergence is $\nabla\cdot\mathbf{v} = 0$ except at the singularity at the origin. The field $\mathbf{v}$ corresponds to a uniform expansion field such as that of a rubber sheet in the process of being stretched. Examine any point P and the neighbouring arrows: some point towards P, but the arrows pointing away from$P$ are bigger so there is a net "pointing away" and the divergence is everywhere positive. The velocity field $\mathbf{v}$ corresponds to the sort of flow obtained when you turn on a tap and pour water down onto a flat surface: immediately under the tap there is a source of water (the singularity), but everywhere else it just spreads with a velocity that slows the further away from the source. The divergence being zero is a borderline case so it is hard to be sure by visual inspection of, for example, the above vector field.

At least this must happen if the fluid cannot expand or compress. For example, water is incompressible (very nearly) and so its velocity field is divergence free, that is it satisfies $\nabla\cdot\mathbf{v} = 0$. Air can expand or compress and so may have some divergence. However, compressible effects in air only become significant for very fast flows — those which are a significant fraction of the speed of sound. Thus, in many situations of interest the flow of air is also effectively incompressible.

Another graphical way of interpreting divergence can be given through considering a small region with imaginary elastic boundaries which are oriented so that its two opposite sides are parallel to the local velocity vectors and two others are perpendicular to them. In both cases considered above such regions are elements of circular rings with the centre at the origin. Since the boundaries of such elements are "elastic" the region will be deformed by the flowing fluid. In the first of the discussed plots the velocity increases with the distance from the origin so that the outer boundary moves faster and the region stretches in the radial direction. It also stretches circumferentially and thus its volume (area) increases. Therefore the divergence is positive here.

The situation is different for the second plot. Although the region stretches in the circumferential direction, the boundary which is closer to the origin moves faster trying to "catch up" with the opposite side of the region and making it thinner. Thinning of the region in the radial direction and stretching it in the circumferential direction have opposite effects on the volume (area) of the region. Therefore it is impossible to say from the graph alone what effect would prevail. Only algebraic calculations of the divergence can determine the sign of the divergence in this case or say that it is zero.

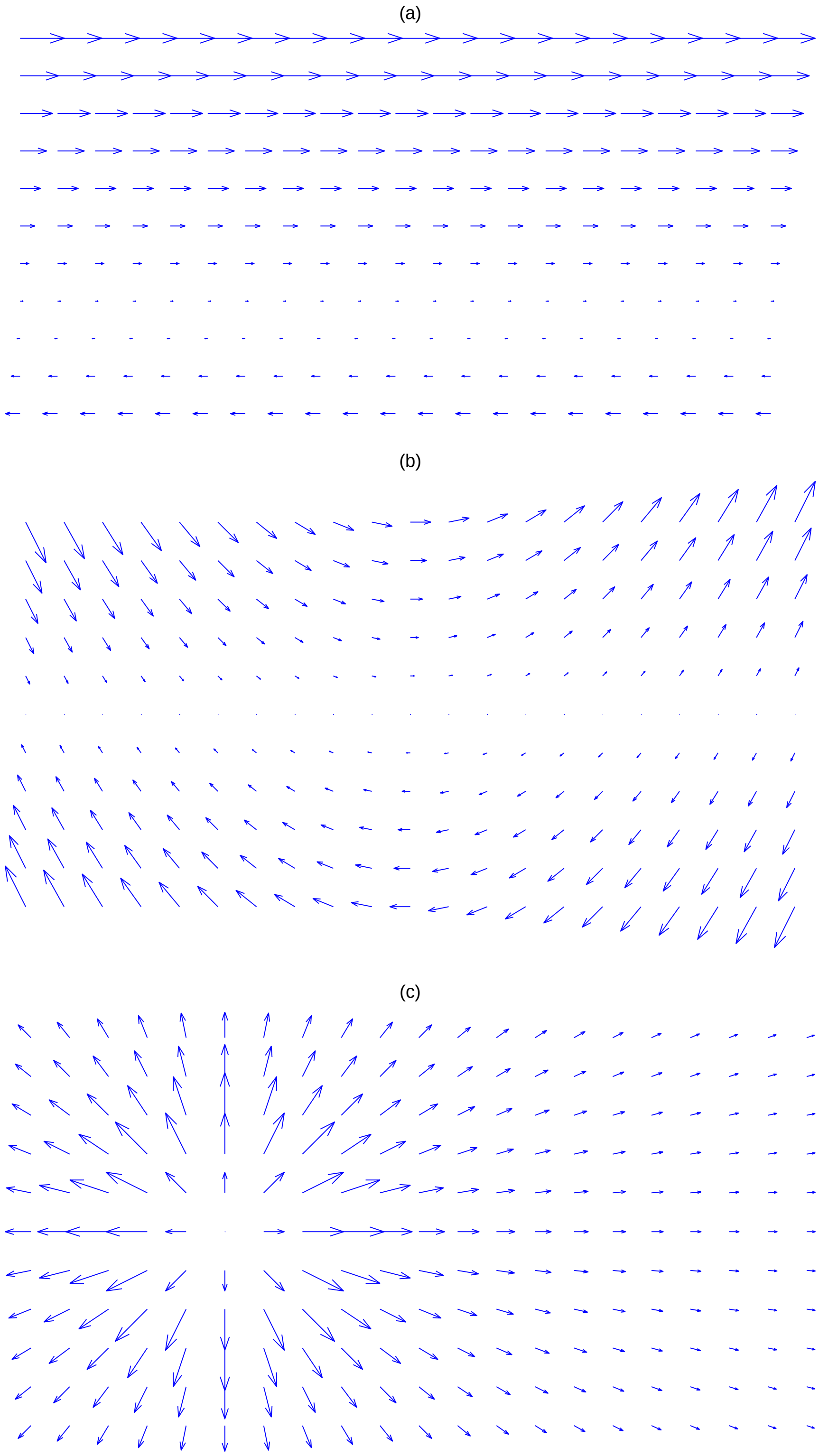
Therefore we conclude, that the physical property characterised by the divergence of a velocity field is the ability of fluid to change its volume and a flow of incompressible fluid is always characterised by zero divergence.

Activity 1.I → Do the following exercises. Send in to the lecturer for feedback solutions for Problems 2 and 6 of Exercise 1.16, and Exercise 1.17(b) below.

1.3.2 Divergent exercises

Ex. 1.16: Which of the vector fields in Problems 1–7 of Problem Set 9.8 [Kre06, p. 413] could describe the flow of an incompressible fluid?

Ex. 1.17: For the following vector fields, sketch regions where you think the divergence is positive, negative, and approximately zero. Consider a variety of points, and for any one point look at the net effect of vectors in its immediate neighbourhood.



1.3.3 Continuity equation

A Biologist, Physicist and Mathematician were in a bar. They watch two people enter a house across the street. A little later, they see three people leave the house. The Biologist

> says, "They must have reproduced." The Physicist says, "We must have misinterpreted the initial input." The Mathematician says, "If one more person enters the house, there will be no one inside."

Thus one application of the divergence, very important for us in this course, is its appearance in the *conservation of mass* which is one of the most important empirical laws of physics: matter cannot disappear without a trace or appear from "nowhere". This fundamental law is applicable to solid bodies, gases and liquids. It is well known that all materials consist of very small particles — *molecules*. The sizes of molecules are so small that about 40 000 molecules of water can be easily placed on the very end of a sewing needle. Intermolecular forces (typically of electromagnetic nature) hold molecules close together so that even on the scale of a fraction of a millimetre the individual properties of a molecule are not felt: fluid behaves as continuous matter, i.e. its properties (e.g. density, viscosity) vary smoothly in space and time. Therefore for a majority of flow situations fluid is considered as a continuum. For this reason an equation describing fluid as a continuous medium is called the continuity equation. It is a mathematical interpretation of the mass conservation principle. For a *compressible fluid* such as air the *continuity equation* is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \mathbf{v}) = 0\,, \tag{1.5}$$

whereas for an *incompressible fluid* such as water, the density is constant and the continuity equation (1.5) reduces to

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0\,. \tag{1.6}$$

This simply asserts that the velocity field of the incompressible fluid flow must be divergence free.

Activity 1.J → Watch the beginning of the video "Fluid Dynamics of Drag, Part II" [FDO] ([CD08], Continuity Concept). Understand the concepts of fluid particle and continuous media. Do a selection of problems from Problem Set 9.8 [Kre06, pp. 413–414]. Send in to the lecturer for feedback solutions for Problems 4 and 13(a,b) from [Kre06, p. 413] and Exercises 1.18 (b), 1.19 and 1.20 below.

See the introductory discussion of the divergence at
`http://www.math.montana.edu/frankw/ccp/multiworld/divcurl/prologue/learn.html`
and at
`http://www.math.montana.edu/frankw/ccp/multiworld/divcurl/divtwothree/learn.html`.

1.3.4 Continuity exercises

Ex. 1.18: Consider the vector fields on the previous page which represent velocity fields of a *compressible fluid*. Mark regions where you expect

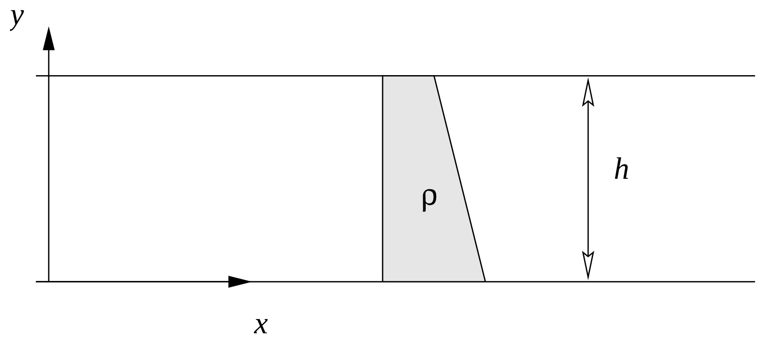


Figure 1.3: Sketch of **Ex. 1.20**

the density to be increasing in time and where it would be decreasing. Assume that initially the density is uniform throughout the whole region.

Ex. 1.19: Are terms "incompressible fluid" and "constant density fluid" equivalent? Do the expressions "incompressible flow of compressible fluid" and "compressible flow of incompressible fluid" make sense? Justify your answers.

Ex. 1.20: Discuss what form of continuity equation has to be used in a variable salinity two-dimensional water layer with density initially given by

$$\rho = \rho_0 \left(1 - a\frac{y}{h}\right) , \quad 0 < a < 1$$

(see Figure 1.3).

1.4 Stream function and its geometrical and physical meanings

Section 1.1 has introduced *streamlines*, i.e. the lines which are parallel to the local fluid velocity vector at every point. Geometrical arguments were used to derive the algebraic expressions defining streamlines. In this section we will see how the continuity equation can be used to derive an equation for streamlines for many fluid flows.

Objectives:

ψ is the Greek letter "psi"

- to define stream function ψ for two-dimensional incompressible flows;
- to show that $\psi = \text{const.}$ is an equation for a streamline;
- to understand the physical meaning of stream function.

Reading 1.K → [Kre06], pp. 405–409 and 761–766

Consider a two dimensional flow of incompressible fluid ($\rho = \text{const.}$) with velocity $\mathbf{v} = (u, v)$. The continuity equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \,. \tag{1.7}$$

Let

$$u = \frac{\partial \psi}{\partial y}\,, \quad v = -\frac{\partial \psi}{\partial x}\,. \tag{1.8}$$

Substitute (1.8) into (1.7) to obtain

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \equiv 0\,. \tag{1.9}$$

Function $\psi(x, y)$ which defines the flow velocity via relations (1.8) and satisfies the continuity equation identically is called a *stream function*. The above derivations might seem very straightforward, but this simplicity is superficial. Indeed how can one "guess" that definition (1.8) works? Let us see if the continuity equation (1.7) itself provides any hints. Starting with minimum assumptions we should structure our deduction in a way that gradually eliminates any ambiguity leaving us with a well-defined solution.

Analyse the structure of the continuity equation (1.7) first. It has *two* terms whose sum is equal to zero. Symbolically it can be written as $A+B = 0$. This means that with *no ambiguity* the two terms in the continuity equation (1.7) must have the same magnitude, but be of opposite signs: $A = -B$. Now look at the structure of the individual terms A and B. They are given by derivatives of different independent functions with respect to different independent variables. They can only be equal to each other in magnitude everywhere if somehow they are reduced to the same function, i.e. both A and B must be functions of both variables x and y. Still this is too ambiguous.

It is known that for "good" twice differentiable functions of several variables the order of differentiation is not important: $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$. Thus both $|A(x, y)|$ and $|B(x, y)|$ must be equal to a mixed derivative of some function of two variables:

$$A = -B = \frac{\partial \psi(x, y)}{\partial x \partial y} = \frac{\partial \psi(x, y)}{\partial y \partial x}\,.$$

Now write $A + B = 0$ using the above expressions in the form which is closest to the original continuity equation (1.7)

$$\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0\,.$$

The term-by-term comparison with equation (1.7) unambiguously results in the definition (1.8) of a stream function for two-dimensional flow of incompressible fluid. Note that not every form of the continuity equation and therefore not every flow allows introduction of the stream function: only flows for which the above procedure results in the unambiguous definition of ψ can be described using the stream function, see **Stream Exercises** in the end of this section.

What is the significance of the stream function apart from the fact that it eliminates the continuity equation by satisfying it identically? Take the gradient of ψ

$$\boldsymbol{\nabla}\psi = \frac{\partial \psi}{\partial x}\mathbf{i} + \frac{\partial \psi}{\partial y}\mathbf{j} = -v\mathbf{i} + u\mathbf{j}\,. \tag{1.10}$$

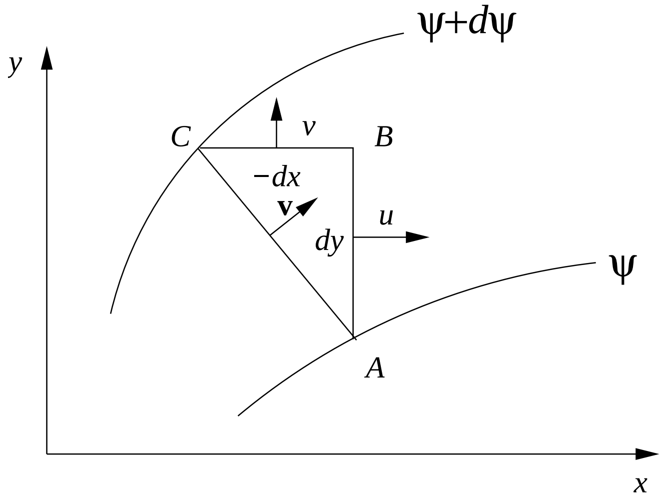


Figure 1.4: Physical meaning of stream function.

Consider a dot product

$$\mathbf{v}\cdot\boldsymbol{\nabla}\psi = u(-v) + vu \equiv 0\,. \tag{1.11}$$

This means that $\mathbf{v}\perp\boldsymbol{\nabla}\psi$. On the other hand $\boldsymbol{\nabla}\psi$ is orthogonal to the *level curves* $\psi =$ const., see Section 1.2. We conclude then that velocity $\mathbf{v}$ is tangent to the level curves and thus the *geometrical meaning of the stream function* is that its level curves $\psi =$ const. are the streamlines for the velocity field $\mathbf{v}$ according to the definition introduced in Section 1.1.

In order to understand the *physical meaning of a stream function* consider two close streamlines as shown in Figure 1.4. As follows from the above discussion the value of stream function is constant along each of the streamlines. Since streamlines are tangent to the local velocity fluid cannot cross them and thus the incompressible *fluid flux* must be the same through any cross-section connecting the streamlines, and be equal to, for example, that through the section shown by line ABC. Then the elementary fluid flux is computed as

$$dQ_{AC} = dQ_{AB} + dQ_{BC} = u\,dy + v\,(-dx) = \frac{\partial\psi}{\partial y}\,dy + \frac{\partial\psi}{\partial x}\,dx = d\psi\,. \tag{1.12}$$

Here segment BC corresponds to $-dx$ because it is travelled in the direction opposite to the positive direction of x. The total flux through the "*flow pipe*" between two streamlines separated by a finite distance then is

$$\int_A^C dQ_{AC} = \int_A^C d\psi = \psi(C) - \psi(A)\,. \tag{1.13}$$

Thus the difference between values of the stream function corresponding to different streamlines gives *volumetric flow rate* of incompressible fluid between these streamlines. This is the physical meaning of stream function introduced for a flow of incompressible fluid.

Example 1.21: Find streamfunction for a velocity field given by $\mathbf{v} = (u, v) = (-x, y)$.

Firstly, check whether the given velocity field is incompressible:

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial(-x)}{\partial x} + \frac{\partial y}{\partial y} = -1 + 1 = 0\,.$$

Since the divergence is zero the flow is incompressible and by definition (1.8)

$$\frac{\partial \psi}{\partial y} = u = -x\,, \quad \frac{\partial \psi}{\partial x} = -v = -y\,.$$

Integrating the first equation with respect to y we obtain

$$\psi = -xy + f(x)\,.$$

Integrating the second equation in x we obtain

$$\psi = -xy + g(y)\,.$$

Since both answers should give rise to the same function we must enforce that

$$\psi = -xy + f(x) = -xy + g(y)\,.$$

By inspection we conclude that this can only be satisfied if $f(x) = g(y) = C$ so that

$$\psi = -xy + C\,.$$

The numerical value of the integration constant C normally is not important because only the difference in the streamfunction values has physical meaning. The streamlines are obtained by setting the streamfunction to constant:

$$\psi = -xy + C = C_1 \text{ or } y = \frac{C - C_1}{x} = \frac{K}{x}\,.$$

Plot a few of these streamlines to see that they represent a flow in the corner bounded by the coordinate axes.

Activity 1.L → Watch the video "Flow Visualization" [FV] ([CD08], Flow Visualisation). Understand how streamlines are obtained for experimental flow fields. Send in to the lecturer for feedback solutions for Exercises 1.22–1.27.

1.4.1 Stream exercises

Ex. 1.22: Can stream function be introduced for compressible flows? If so, what limitations should be imposed on such flows and what could the relation between the stream function and velocity then be?

Ex. 1.23: Can stream function be introduced for three-dimensional flows?

Ex. 1.24: If the stream function could be introduced for compressible flow, what would its physical meaning be?

Ex. 1.25: If the stream function could be introduced for compressible flow, what would its geometrical meaning be?

Ex. 1.26: Derive an expression for the stream function ψ for a uniform flow $\mathbf{v} = (u, v) = (V\cos\alpha, V\sin\alpha)$, where V and α are constants. What is the streamline equation for this flow? Sketch a few streamlines and a few equi-potential lines (see **Ex. 1.12**) on the same plot enforcing equal scales in both horizontal and vertical directions (in MATLAB, use command `axis('equal')`). What is the angle between the two types of lines? Justify your answer algebraically.

Ex. 1.27: Find the flow rate between two streamlines passing through points (x_1, y_1) and (x_2, y_2) in **Ex. 1.26**.

1.5 Vorticity is the curl of velocity

Q: What do you get when you cross a tsetse fly with a mountain climber?
A: Nothing, you can't cross a vector with a scalar.

The third and final fundamental differentiation operator in vector calculus is that of the *curl*. Computed in Cartesian coordinates by the formula, that if $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ then

$\nabla\times$ is pronounced "curl".

$$\begin{aligned}\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\ &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}\,,\end{aligned} \tag{1.14}$$

it may be simply applied to determine, for example, that

$$\nabla\times(-x\mathbf{i} + y\mathbf{j}) = 0\mathbf{i} + 0\mathbf{j} + \left(\frac{\partial(y)}{\partial x} - \frac{\partial(-x)}{\partial y}\right)\mathbf{k} = \mathbf{0}\,.$$

Here the curl just happens to be everywhere the zero vector. In general, the curl acts upon a vector field to result in another vector field.

Objectives:

- to refresh and use your knowledge of the curl;
- to introduce the *vorticity* as the the curl of a velocity field;
- to introduce irrotational flow and the necessity of the velocity potential to satisfy Laplace's equation;

Reading 1.M $\rightarrow$ Study Section 9.9 in Kreyszig [Kre06, pp. 414–416].

Activity 1.N $\rightarrow$ Do a selection of problems from Problem Set 9.9 [Kre06, p. 416]. Send in to the lecturer for feedback solutions for Problems 7, 12 and 20.

1.5.1 The vorticity

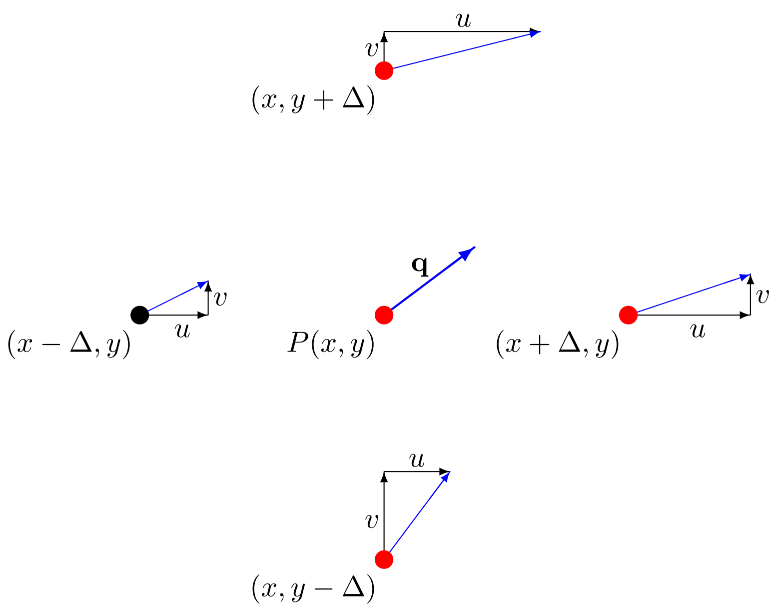
> Big whorls have little whorls
> to feed off their vorticity;
> and little whorls have lesser whorls,
> and so on to viscosity.
>
> *Lewis Fry Richardson*

In fluid dynamics, the curl of the velocity field is called the *vorticity*, usually denoted by $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$. For example, confirm that the velocity field $\mathbf{v} = yz\mathbf{i} + 3zx\mathbf{j} + z\mathbf{k}$ has the associated vorticity field $\boldsymbol{\omega} = -3x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$.

Many decades ago the vorticity was also called the rotation. Although this term itself has lost favour and gone out of use, it still lives on in the term *irrotational* which is used for a vector field with zero curl and hence in particular to refer to a fluid flow for which the vorticity is zero everywhere. For example, as shown earlier, $\boldsymbol{\nabla} \times (-x\mathbf{i} + y\mathbf{j}) = \mathbf{0}$ and so $\mathbf{v} = -x\mathbf{i} + y\mathbf{j}$ is an irrotational velocity field.

In Australia, to dry clothes after washing we often hang them outside upon a rotary clothes hoist. Being free to revolve, whenever the wind is strong enough it will do so. Why? Because wind is generally uneven, especially as it eddies and swirls about buildings and trees, and these variations in the wind cause a net rotational force (a torque) about the axis of the rotary clothes line. The vorticity quantifies such local swirls and eddies in the velocity field of the wind, as the curl does in general for other vector fields.

Very loosely speaking, the vorticity — the curl of a vector field — at a point P measures the rate that the vector field locally "rotates" about P. See this crudely via the following argument based on that used for the divergence. Choose any point $P(x, y)$ and again consider a velocity field $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$ at four neighbouring points in two dimensions as shown below.



What do we mean by the rate a vector field "rotates" about a point P? It is the average around neighbouring points of the component of $\mathbf{v}$ directed at right angles to the central point P and hence tending to twist the vector field about the pivotal point P. Here:

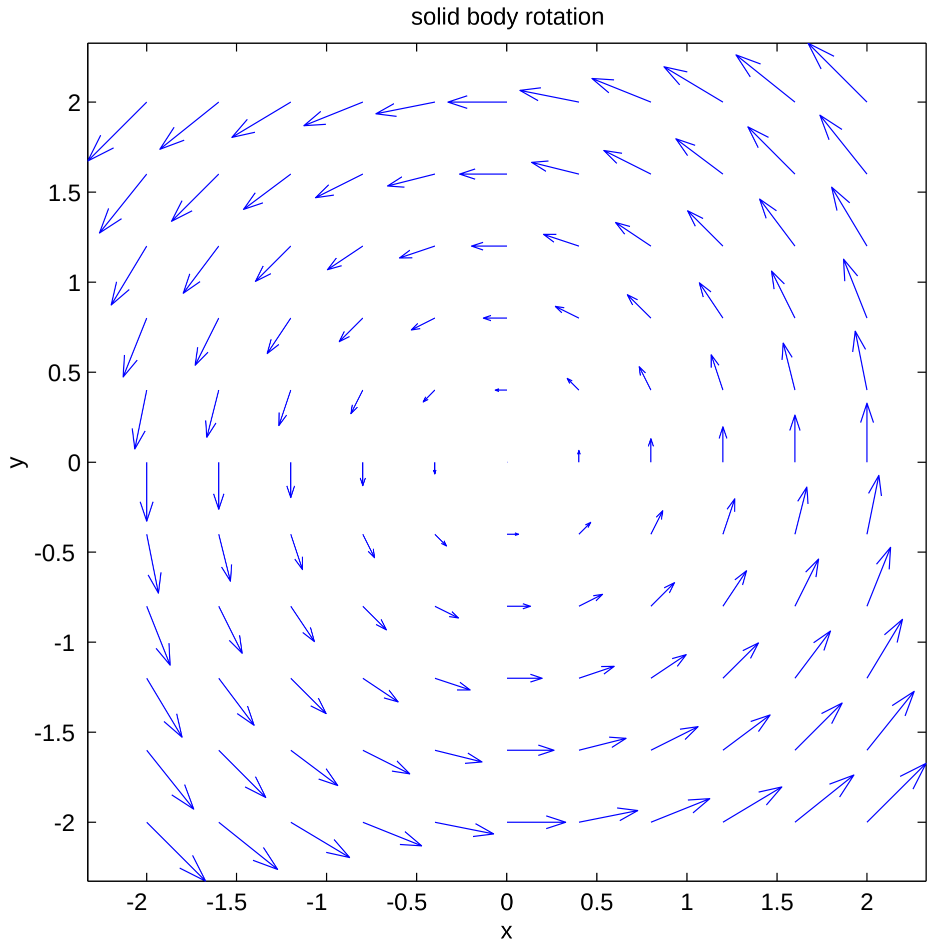
- to the right at $(x+\Delta, y)$ the anti-clockwise rotational component is $v(x+\Delta, y)$ as the u component is directed neither one way nor the other around P;
- below at $(x, y-\Delta)$ the anti-clockwise component is $u(x, y-\Delta)$ as here it is the v component that is pointing neither one way nor the other around P;
- to the left at $(x-\Delta, y)$ the component is $-v(x-\Delta, y)$ (minus because positive v points clockwise around P);
- above at $(x, y+\Delta)$ the component is $-u(x, y+\Delta)$ (minus because positive u points clockwise around P).

But note that these components all act at a distance Δ away from P, so that the average "rate" they "rotate" about P is given by the above components divided by the length of the rotation arm, namely Δ. Thus the average rate of rotation caused by these components is

$$\begin{aligned}
&\frac{1}{4\Delta}\left[v(x+\Delta, y) - u(x, y+\Delta) - v(x-\Delta, y) + u(x, y-\Delta)\right] \\
&= \frac{1}{2}\left[\frac{v(x+\Delta, y) - v(x-\Delta, y)}{2\Delta} - \frac{u(x, y+\Delta) - u(x, y-\Delta)}{2\Delta}\right] \\
&\approx \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] \quad \text{by definition of the derivative} \\
&= \frac{1}{2}(\text{the } \mathbf{k} \text{ component of } \boldsymbol{\nabla} \times \mathbf{v})\,.
\end{aligned}$$

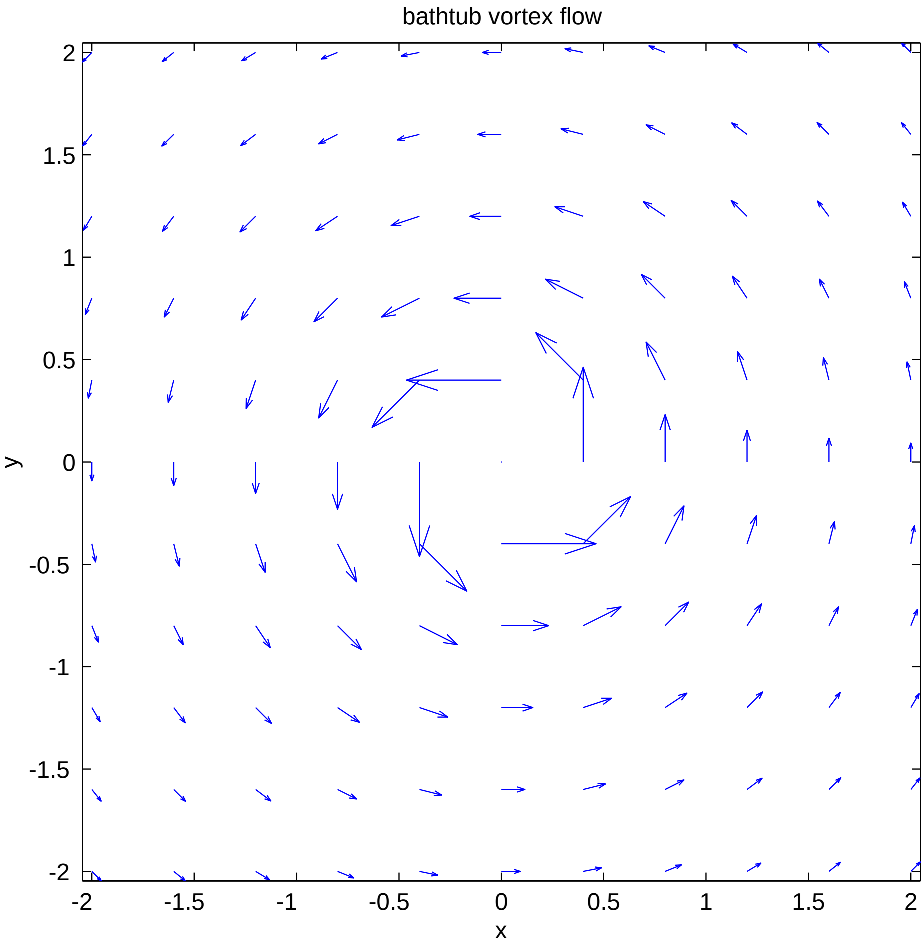
This demonstrates that the curl of a vector field at any point is proportional to the *local* rate of rotation about that point inherent in the vector field.

Two examples illustrate this. Contrast the vector field of a solid body rotation $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$ (rotating at a rate of 1 radian per second) shown below




```
[x,y]=meshgrid(-2:0.4:2);
quiver(x,y,-y,x)
```

for which the vorticity is everywhere $\boldsymbol{\nabla} \times \mathbf{v} = 2\mathbf{k}$, with the vortex vector field $\mathbf{v} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)$ shown here




```
[x,y]=meshgrid(-2:0.4:2);
u=-y./(x.^2+y.^2);
v= x./(x.^2+y.^2);
quiver(x,y,u,v)
```

for which the vorticity is $\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}$ except at the singularity at the origin. In both of these fields the fluid particles travel in circles around the origin (show the streamlines are circular). But in any locale, a particle of the solid body revolution rotates its orientation, whereas a particle in the vortex field does not.

Try these experiments for yourself — computer simulations can be done but are just not the same as the real thing.

- The field $\mathbf{v}$ is that of solid body rotation such as that seen in a revolving record turntable. Imagine putting a small lego block on such a turntable as it turns. The lego block, as it travels around the central spindle, has its orientation rotated. This uniform rotation in the orientation corresponds to the uniform vorticity of $\mathbf{v}$.
- In contrast the velocity field $\mathbf{v}$ corresponds to the sort of flow obtained after you place a layer of water into a washbasin, stir the water gently into a slow and smooth rotation, then pull the plug to let the water drain. (The washbasin should have a flat bottom and a centrally located drain hole.) Soon after pulling the plug the flow organises itself into a flow field roughly like $\mathbf{v}$: close to the drain the

water swirls around rapidly, but the speed of travel about the drain slows the further away from the drain, just as in the picture. Now float a lego block on the surface of the water. See that as it travels with the water about the central drain, *the orientation of the block does not change* (at least not as much as in solid body rotation). This lack of local rotation of the orientation corresponds to the zero vorticity of this $\mathbf{v}$.

See the pictures at http://www.sci.usq.edu.au/staff/robertsa/basinvortex/sequence.html.

It is the *local* rotation in the velocity field that the curl quantifies as vorticity: everywhere uniform in the case of solid body rotation; and none in the ideal bathtub vortex flow.

Finally, another example of irrotational motion, though not in fluids, occurs at carnivals. Consider a Ferris wheel rotating majestically about its axis. People going for a ride on the Ferris wheel sit in chairs that always hangs downwards. As they revolve about the axis of the Ferris wheel, the people and chairs maintain the same orientation. Thus they do not rotate — they undergo irrotational motion on the revolving Ferris wheel.

1.5.2 Irrotational flows lead to Laplace's equation

The vorticity is an important part of fluid dynamics. However, in some cases it can be neglected. For example, in the bathtub vortex, the vorticity is concentrated in a small region of the flow leaving large regions of the flow free of vorticity. We call such flows irrotational flows. In these regions $\boldsymbol{\nabla}\times\mathbf{v}=\mathbf{0}$. Recall one of the basic vector identities, that for any scalar function φ:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\varphi)=\mathbf{0}\,.$$

Thus one way to assure an irrotational flow is to confine attention to velocity fields obtained as the gradient of a velocity potential: $\mathbf{v}=\boldsymbol{\nabla}\varphi$. The converse, that all irrotational velocity fields must indeed have a velocity potential is also true but is somewhat harder to prove. The proof requires Stokes' theorem which we meet later (§2.5).

Thus the flows of primary interest, irrotational flows, may be discussed either using the velocity field, $\mathbf{v}$, or using the velocity potential, φ. But usually we restrict attention to incompressible flows which also satisfy the continuity equation $\boldsymbol{\nabla}\cdot\mathbf{v}=0$. Thus the only velocity potentials of interest to us are those that satisfy

∇^2 is pronounced "del-squared".

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\varphi)=\nabla^2\varphi=0\,.$$

Namely, the velocity potential of an irrotational and incompressible flow must satisfy Laplace's equation[2]

$$\nabla^2\varphi=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0\,. \tag{1.15}$$

[2] Pierre Simon Laplace (1749–1827) has been called the "Newton of France." He worked on astronomy, fluids, tides, heat transfer, surface tension, electric theory and probability.

For example, the velocity field $\mathbf{v} = -x\mathbf{i} + y\mathbf{j}$ has velocity potential $\varphi = -\frac{1}{2}x^2 + \frac{1}{2}y^2$ for which

$$\nabla^2\varphi = -1 + 1 = 0\,.$$

It follows from the above discussion that every solution of the Laplace's equation can be interpreted as a velocity potential corresponding to some fluid flow. This is a remarkable conclusion: there is an enormous variety of fluid flows around us and many of them are described by a solution of a single linear partial differential equation which has the same canonical form no matter how complicated the flow it describes is. Then how does this equation "know" what particular flow should it describe? The answer is through the boundary conditions. The Laplace's equation is a second order elliptic linear partial differential equation (review MAT3105) and thus it requires specifying the boundary conditions along all surfaces bounding the flow region. Below we discuss the two most typical situations.

Flow over a resting solid body. No fluid can penetrate through the surface of a solid body. This means that the velocity component perpendicular to the body surface must be zero. Let $\mathbf{n}$ be the unit normal vector to the body surface. Then the normal velocity component is simply given by a projection v_n of a full velocity $\mathbf{v}$ onto $\mathbf{n}$. Since $\mathbf{v} = \boldsymbol{\nabla}\varphi$

$$v_n = \mathbf{n}\cdot\mathbf{v} = \mathbf{n}\cdot\boldsymbol{\nabla}\varphi = \frac{\partial\varphi}{\partial\mathbf{n}} = 0\,,$$

where the directional derivative along the unit normal vector $\mathbf{n}$ was introduced.

The above condition is valid at any solid surface, but what if we consider an unbounded flow, e.g. the flow around an oil rig in an ocean? Surely far enough from the oil rig the flow will not "know" anything about the presence of the oil rig and will be defined by some other conditions, say, by the ocean current with a defined velocity $\mathbf{v}_\infty$. Then the boundary conditions at $x, y, z \to \infty$ become

$$\boldsymbol{\nabla}\varphi = \mathbf{v}_\infty\,,$$

where coordinates x, y and z are measured from the oil rig location.

Solid body moving through a resting fluid. Now imagine a ship moving in the middle of a still ocean. It is clear that far away from a ship the water velocity is 0, i.e.

$$\boldsymbol{\nabla}\varphi = 0\,, \quad x, y, z \to \infty\,.$$

Water cannot penetrate through the ship walls, but now it is pushed by the moving ship. It can slide freely along the ship as here we neglect the water viscosity, but it must move with a ship wall in the direction perpendicular to a solid surface — this is the no-penetration condition for a moving solid body. Thus the boundary condition becomes

$$v_{n\,\text{fluid}} = v_{n\,\text{surface}} \quad \text{or} \quad \frac{\partial\varphi}{\partial\mathbf{n}} = v_{n\,\text{surface}}\,.$$

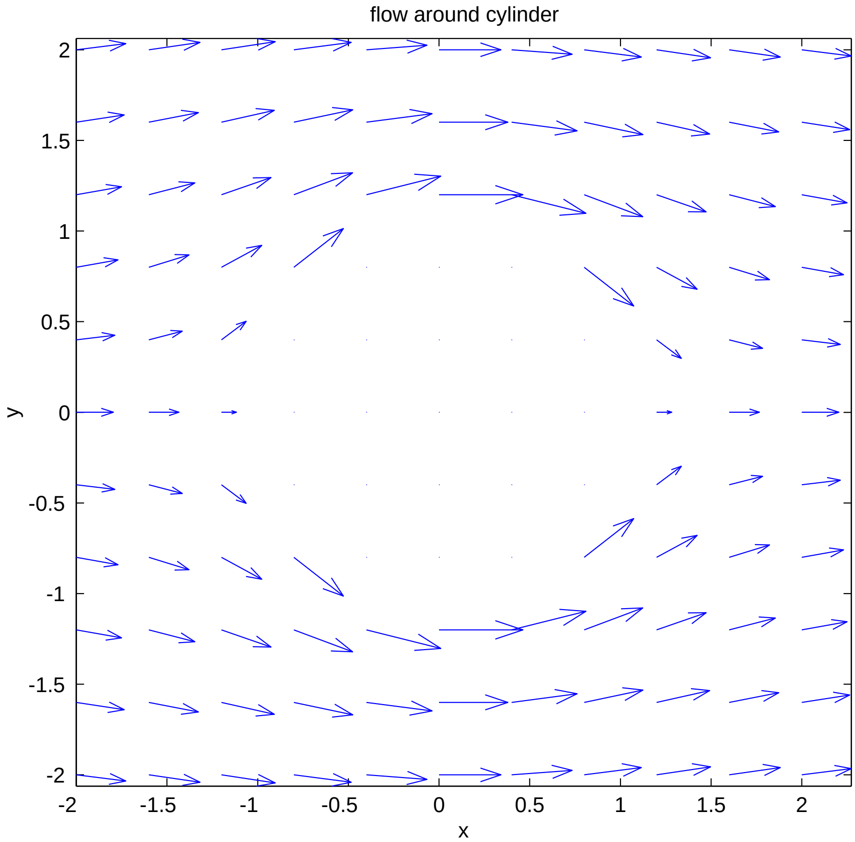
Note: that at large x and y this field is $\mathbf{v} \approx \mathbf{i}$ corresponding to a uniform flow of speed 1 in the x direction.

Example 1.28: Consider the velocity field of a uniform wind blowing around a circular cylinder (aligned along the z-axis):

$$\mathbf{v} = \left(1 + \frac{y^2 - x^2}{(x^2+y^2)^2}\right)\mathbf{i} - \frac{2xy}{(x^2+y^2)^2}\mathbf{j} = \left(1 + \frac{y^2 - x^2}{r^4}\right)\mathbf{i} - \frac{2xy}{r^4}\mathbf{j}\,,$$

using r as a shorthand to denote $\sqrt{x^2+y^2}$. This velocity field is shown below

The inside of the cylinder is omitted by setting the relevant data to nan or "not a number".




```
[x,y]=meshgrid(-2:0.4:2);
u=1+(y.^2-x.^2)./(x.^2+y.^2).^2;
v=      -2*x.*y./(x.^2+y.^2).^2;
incyl=find(x.^2+y.^2<1);
u(incyl)=repmat(nan,size(incyl));
v(incyl)=u(incyl);
quiver(x,y,u,v)
axis('equal')
```

See how the flow from the left divides smoothly around the cylinder (of radius 1 as it happens) and then rejoins on the right.

Note: $\partial r/\partial x = x/r$ and similarly for y.

- The vorticity of the flow is

$$\begin{aligned}
\nabla\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1+\frac{y^2-x^2}{r^4} & -\frac{2xy}{r^4} & 0 \end{vmatrix} \\
&= \mathbf{i}0+\mathbf{j}0+\mathbf{k}\left[\frac{\partial}{\partial x}\left(-\frac{2xy}{r^4}\right)-\frac{\partial}{\partial y}\left(1+\frac{y^2-x^2}{r^4}\right)\right] \\
&= \mathbf{k}\left[-\frac{2y}{r^4}+\frac{8x^2y}{r^6}-\frac{2y}{r^4}+\frac{4y(y^2-x^2)}{r^6}\right] \\
&= \mathbf{0}\,,
\end{aligned}$$

and is thus irrotational.

- The divergence of the flow is

$$\begin{aligned}
\nabla\cdot\mathbf{v} &= \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \\
&= \frac{-2x}{r^4}-\frac{4x(y^2-x^2)}{r^6}-\frac{2x}{r^4}+\frac{8xy^2}{r^6} \\
&= 0\,,
\end{aligned}$$

and is thus incompressible.

- A velocity potential, φ, must then exist for this flow.
  - The defining property of φ is: $\frac{\partial\varphi}{\partial x}=u=1+\frac{y^2-x^2}{r^4}$, $\frac{\partial\varphi}{\partial y}=v=-\frac{2xy}{r^4}$, and $\frac{\partial\varphi}{\partial z}=w=0$.
  - The last says that φ is independent of z.
  - Recalling that $\frac{\partial r}{\partial y}=\frac{y}{r}$, the v equation may be written as

$$\frac{\partial\varphi}{\partial y}=-\frac{2x}{r^3}\frac{y}{r}=-\frac{2x}{r^3}\frac{\partial r}{\partial y}\,,$$

    and hence integrated with respect to y to determine $\varphi=\frac{x}{r^2}+f(x)$ for some function f.
  - Finally, the u equation is used to determine that $\frac{\partial f}{\partial x}=1$, thus $f=x$ (to within an irrelevant arbitrary constant), and hence

$$\varphi=x+\frac{x}{r^2}$$

    is the required velocity potential for this flow around a circular cylinder.

Activity 1.O → Do the following exercises. Send in to the lecturer for feedback solutions for Problem 4 of Exercise 1.30, and Exercises 1.31(a) and (b) below.

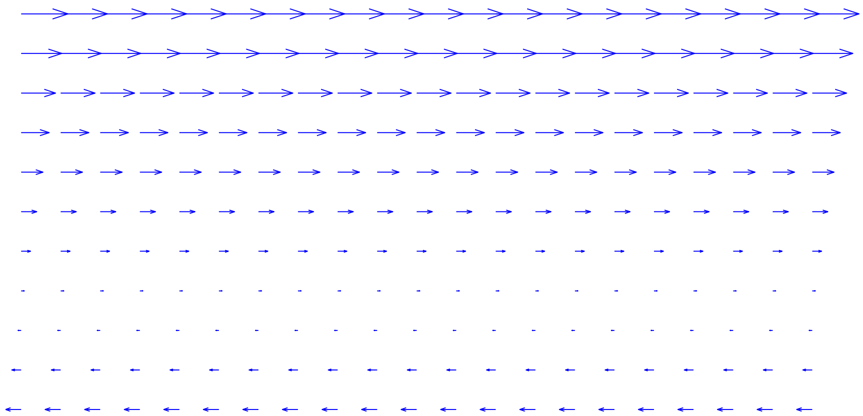
1.5.3 Curly exercises

Ex. 1.29: Which of the potentials that you found for Problems 39–41 in Problem Set 9.7 [Kre06, p.409–410] satisfy Laplace's equation and thus are the velocity potential for some irrotational fluid flow?

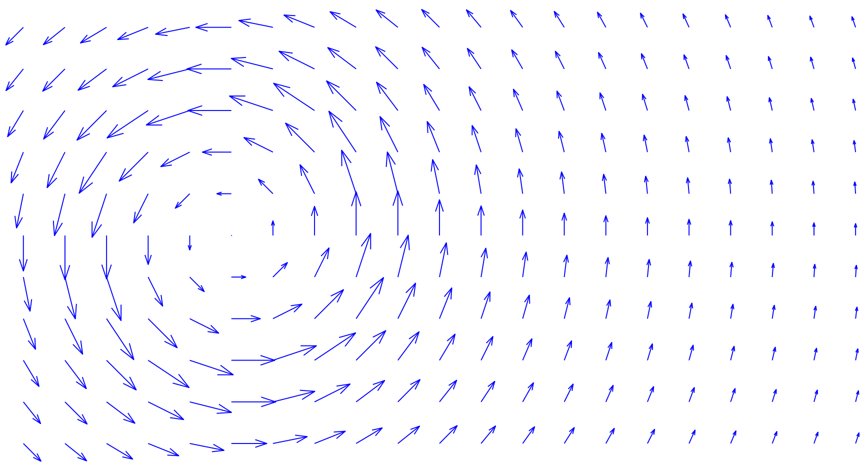
Ex. 1.30: Which of the vector fields in Problems 1–14 of Problem Set 9.9 [Kre06, p. 416] could describe the irrotational flow of an incompressible fluid. For those that do, find a velocity potential.

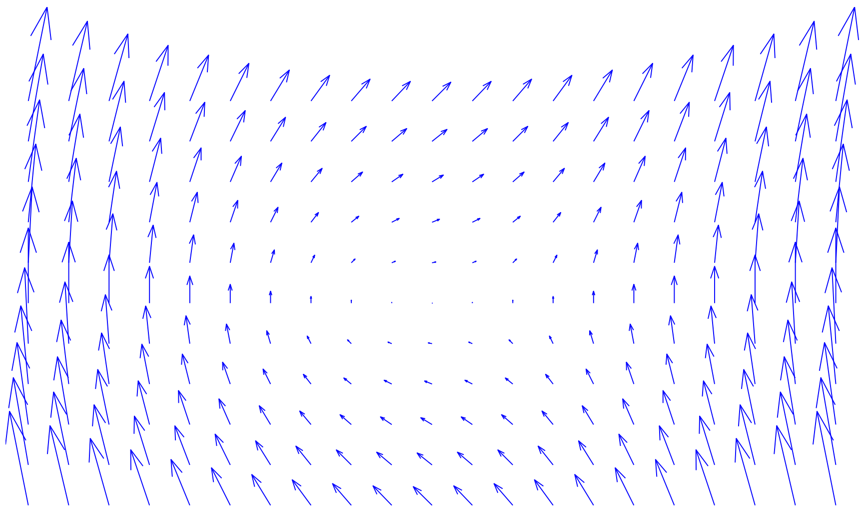
Ex. 1.31: For the following velocity fields, indicate regions where you expect the vorticity is positive (the local $\mathbf{v}$ rotates anti-clockwise), the regions it is negative ($\mathbf{v}$ locally rotates clockwise), and regions where it is approximately zero.

(a)



(b)





1.6 Conservation of momentum leads to the Euler's equation

Nature and Nature's laws lay hid in night:
God said "Let Newton be", and all was light.
Alexander Pope's epitaph for Newton

The continuity equation (1.5) derived in Kreyszig [Kre06, §9.8] is based upon the principle of conservation of mass. In fluid and solid mechanics another fundamental principle is that *momentum is conserved.* In this section we use this conservation principle to derive Euler's equation (pronounced "Oiler") for fluid flow. Euler's equation is the continuum analogue of Newton's second law of motion that the net force $\mathbf{F}$ acting on a body is equal to the product of the body mass m and its acceleration $\mathbf{a}$: $\mathbf{F} = m\mathbf{a}$. In the special but useful case of incompressible and irrotational flow, Euler's equation is integrated explicitly to determine the pressure field via what is called Bernoulli's equation.

Objectives:

- to derive Euler's equation from the principle of conservation of momentum and the Newton's second law;
- to see how the linear wave equation arises in the propagation of water waves;
- to derive and use the Bernoulli equation for the pressure field in an incompressible and irrotational flow;
- to explore some example fluid flows.

The contents of this section are not covered by Kreyszig.

1.6.1 Euler's equation models more general inviscid flows

In Section 1.5.2 we learnt that irrotational inviscid flows in simple domains can be described via the introduction of a. However many flows found in nature are rotational and occur in complicated geometries. Thus a more general mathematical model, the *Euler's equation* is required. In order to derive the Euler equation we start with the Newton's second law $\mathbf{F} = m\mathbf{a}$ which represents a mathematical formulation of the momentum conservation principal. This equation is straightforward to use for the motion of solid bodies whose shape and therefore volume and mass are well-defined. For liquids and gases we reformulate this equation to describe the motion of a fluid particle which does not have a definite shape: we consider the net force acting on a unit volume. The mass of a unit volume of fluid is equal to the fluid density ρ and its acceleration (since the fluid particle moves with the flow) is given by the material derivative of the fluid velocity with respect to time $\mathbf{a} = \frac{D\mathbf{v}}{Dt}$. Then the Newton's second law becomes

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{F}\,, \tag{1.16}$$

where $\mathbf{F}$ is now the net force acting on a unit fluid volume. Equation (1.16) can be written as a system of equations for individual velocity components, see discussion in Section 1.2.1. This system is referred to as the *system of Euler's equations.*

Activity 1.P → Watch the video "Fluid Dynamics of Drag, Part II" [FDO] ([CD08], Continuity Concept). Distinguish between body and surface forces acting on a fluid particle.

Let us consider the nature of the net force $\mathbf{F}$ in detail. Generally speaking it consists of two type of forces: body forces and surface forces. The former act on a fluid particle from a distance, the latter act at contact through the surface of a fluid particle. Examples of body forces are gravity and electromagnetic forces. In this course we will only consider electrically neutral fluids and thus we neglect the action of electromagnetic forces and only consider the gravity force $\mathbf{F}_g = \rho\mathbf{g}$, where $\mathbf{g}$ is the gravity acceleration. An example of a surface force is pressure p which is totally due to the impacts of fluid molecules surrounding the fluid particle with those inside it. Obviously these collisions occur only at the surface of a fluid particle. Note though that the uniform pressure does not have any effect on a fluid particle motion: in this case pressure applied to one of the particle surfaces is exactly the same as that applied to the opposite surface in the opposite direction so that the two contributions cancel each other. Thus pressure force must be due to the pressure difference which is characterised locally by the pressure gradient ∇p. It is also known that the resulting pressure force pushes fluid from high to low pressure regions, i.e. in the direction opposite to the pressure gradient. Summarising we can write that $\mathbf{F}_p = -\nabla p$ and upon division by the density we obtain the most common form of the Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \mathbf{g} \,. \tag{1.17}$$

It applies very generally: to both compressible or incompressible flows; irrotational or not. The only limitation is in the neglect of frictional effects which will be considered in Part II of this course.

In an important case of incompressible flow in a uniform gravitational field equation (1.17) can be simplified by introduction of the so-called *total pressure* $P = p - \rho \mathbf{r} \cdot \mathbf{g}$, where p represents usual pressure due to intermolecular collisions, the product term is the so-called *hydrostatic pressure*[3] and $\mathbf{r} = (x, y, z)$ is the position vector. Indeed the gravity force can be written in the component form as

$$\rho \mathbf{g} = \rho(g_x, g_y, g_z) = \left(\frac{\partial}{\partial x}(\rho x g_x), \frac{\partial}{\partial y}(\rho y g_y), \frac{\partial}{\partial z}(\rho z g_z) \right) = \boldsymbol{\nabla}(\rho \mathbf{r} \cdot \mathbf{g})$$

and then the Euler equation becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = -\frac{1}{\rho} \boldsymbol{\nabla} P \,. \tag{1.18}$$

This form is frequently used in practice whenever considering the hydrostatic pressure is not a focus of a problem.

1.6.2 Water waves

As another example of applying the Euler's equations to a realistic flow problem, we consider small amplitude water waves on a surface of a long and wide channel: you have seen variations of this simple set up many times — flow in shallow creeks, rain water rushing down the gutter, water level oscillations in the gap between the ship and pier walls etc.

Objectives:

- to analyse the problem in order to introduce important simplifying assumptions;
- to justify validity of Euler equations for description of this flow;
- to derive a model equation for the surface wave motion;
- to analyse some of the model solutions.

Consider a flow in a wide channel of uniform depth h. The side walls of the channel might modify the nearby flow, but their influence decays quickly with the distance. Thus the flow near the centre line of the channel does not depend on the transverse coordinate z and a two dimensional consideration of the flow is sufficiently accurate. We limit ourselves to the waves of small amplitude only, i.e. $\eta \ll h$, where $\eta = \eta(x, t)$ is the deviation of the free surface from undisturbed level h as shown in Figure 1.5. In turn

[3] The dot product $\mathbf{r} \cdot \mathbf{g}$ is called the *gravitational potential*.

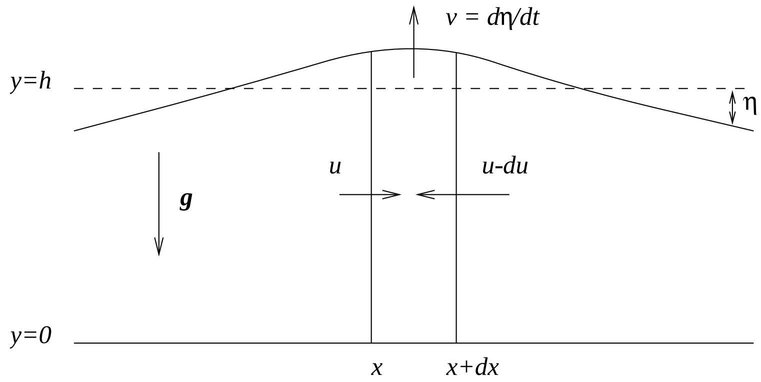


Figure 1.5: Free surface wave.

it means that the motion of interest is "concentrated" in a thin layer near the free surface away from the bottom of the channel. Thus viscous forces which play an important role near the solid boundaries are negligible near the free surface and the Euler equations can be used to describe surface water waves. We also assume that the waves are sufficiently smooth and have a long wavelength so that the free surface slope $\frac{\partial \eta}{\partial x}$ is also small.

Consider two vertical planes separated by distance dx as shown in Figure 1.5. The water flux through the left plane is $Q(x) = u(h + \eta(x,t)b$ and through the right wall is $Q(x + dx) = (u - du)(h + \eta(x,t) + \Delta\eta)b$, where b is the width of the channel and $\Delta\eta \ll \eta$ is the difference between the free surface elevations at locations x and $x + dx$. Since η itself is assumed to be small $\Delta\eta$ is even smaller and can be neglected. Within the small time interval dt the volume of water between the two planes will change then by a quantity $[Q(x + dx) - Q(x)]dt = -(h + \eta(x,t))b\, du\, dt$. Since water is incompressible and cannot penetrate through the bottom of the channel, the free surface level will have to change in order to accommodate an additional amount of fluid between the planes, i.e. the new fluid level $h + \eta(x, t + dt) = h + \eta(x,t) + d\eta$ has to satisfy the following equation

$$[h + \eta(x,t) + d\eta]b\, dx - [h + \eta(x,t)]\, b\, dx = -(h + \eta(x,t))b\, du\, dt \qquad (1.19)$$

or $d\eta\, dx = -(h+\eta(x,t))\, du\, dt$. Since $\eta(x,t) \ll h$ we neglect it in the right-hand side of this equation and then by rearranging we obtain $d\eta/dt = -h\, du/dx$ at $y = h$ or, taking the limit $\{dx, dt, du\} \to 0$,

$$\frac{\partial \eta}{\partial t} = -h\frac{\partial u}{\partial x}\,. \qquad (1.20)$$

Partial derivatives occur because all functions depend on both x and t. This equation models the motion of the free surface. Note that $\frac{\partial \eta}{\partial t} \approx v(x,h)$, the vertical fluid velocity at the free surface. Since water cannot penetrate through the bottom, $v(x,0) = 0$ and thus we expect that $0 \le v(x,y) \le \left|h\frac{\partial u(x,h)}{\partial x}\right|$, i.e. the vertical velocity component is relatively small throughout the fluid layer. Write the Euler equation for the vertical velocity component

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g\,. \qquad (1.21)$$

Since v is small we obtain to the leading approximation

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -g. \tag{1.22}$$

and, integrating, $p \approx p_0 + \rho g(h + \eta - y)$, where p_0 is the uniform (atmospheric) pressure above the free surface. Now we can write down the Euler equation for the horizontal velocity component as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x}. \tag{1.23}$$

Since we are considering waves of small amplitude we expect that u is also small and, to leading approximation, we can neglect nonlinear terms. Thus we obtain

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}. \tag{1.24}$$

Now differentiate equation (1.20) with respect to t, equation (1.24) with respect to x and eliminate term $\frac{\partial^2 u}{\partial t \partial x}$ to obtain a *wave equation* for the free surface shape η

$$\frac{\partial^2 \eta}{\partial t^2} = c^2\frac{\partial^2 \eta}{\partial x^2}, \tag{1.25}$$

where $c = \sqrt{gh}$ is the *wave propagation speed* (see Exercise 1.33).

Activity 1.Q → Watch the video "Fluid Flows" [FF85], concentrate on the wave kinematics. Which of the two types of waves did we model in this section? Why do you think so? Send in to the lecturer for feedback solutions for Exercises 1.32–1.34.

1.6.3 Wavy exercises

Ex. 1.32: Show that solution of equation (1.25) can be written as $\eta = f_1(x_1) + f_2(x_2)$, where $x_1 = x - ct$ and $x_2 = x + ct$.

Ex. 1.33: Let $\eta(x, 0) = 1$ for $|x| < 5$ and 0 otherwise and $g = h = 10$. Use the results of **Ex. 1.32** and sketch the solutions of equation (1.25) for $t = 0$, $t = 1$ and $t = 2$.

Ex. 1.34: Estimate speed of tsunami tidal wave generated by an earthquake at the bottom of the ocean 5 km below sea level.

1.6.4 Bernoulli's equation determines pressure

Previously we identified that a velocity potential which satisfies Laplace's equation (1.15) has a divergence free gradient and hence gives a velocity field that conserves an incompressible fluid. However, we have yet to see that such a velocity field satisfies the other dynamic equation of fluid mechanics, Euler's equation (1.17). We must check that this is indeed so. In doing this here we derive an equation for the pressure, called Bernoulli's equation[4]. Bernoulli's equation gives the pressure field corresponding to any given velocity field of an incompressible irrotational fluid flow.

Consider each of the three terms in Euler's equation (1.18) in turn. We write each of them as the gradient of something which, remarkably, then enables us to integrate the equation.

- The time derivative $\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial(\boldsymbol{\nabla}\varphi)}{\partial t} = \boldsymbol{\nabla}\left(\frac{\partial\varphi}{\partial t}\right)$ whenever the velocity field may be written as the gradient of a velocity potential, possibly time dependent.
- The advection term $\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\nabla}\left(\frac{1}{2}|\mathbf{v}|^2\right)$. This is straightforward to verify by expanding the right-hand side and appropriately using the irrotational flow identities: $\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$, $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$ and $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, see Exercise 1.36.
- The term $-\frac{1}{\rho}\boldsymbol{\nabla}P = \boldsymbol{\nabla}\left(-\frac{P}{\rho}\right)$ whenever the density is constant.

Thus putting all these terms of Euler's equation on the left-hand side we write it as

$$\boldsymbol{\nabla}\left(\frac{\partial\varphi}{\partial t}\right) + \boldsymbol{\nabla}\left(\frac{1}{2}|\mathbf{v}|^2\right) + \boldsymbol{\nabla}\left(\frac{P}{\rho}\right) = \mathbf{0}\,.$$

But the sum of gradients is the gradient of the sum, thus

$$\boldsymbol{\nabla}\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\mathbf{v}|^2 + \frac{P}{\rho}\right) = \mathbf{0}\,.$$

Now, here on the left we have the gradient of some complicated function of the fluid flow. On the right we see that this gradient has to be everywhere zero. But the only function whose gradient is *everywhere* zero is a constant function — constant in space though it may vary in time, call it $C(t)$. Thus we integrate the last equation to obtain the *Bernoulli's equation*

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\mathbf{v}|^2 + \frac{P}{\rho} = C(t)\,. \tag{1.26}$$

In this derivation we required an incompressible fluid, i.e. constant ρ, an irrotational flow, we did not allow the fluid to be added or subtracted through any sinks or sources and we assumed that no energy is supplied to fluid via heating/cooling or any mechanical action. Thus the above Bernoulli's equation is valid under these conditions only. If the flow is steady, the Bernoulli equation becomes

$$\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\rho} - \mathbf{r}\cdot\mathbf{g} = \text{const.}$$

If in addition the flow direction is perpendicular to the gravity

$$\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\rho} = \text{const.}$$

[4] Daniel Bernoulli (1700–82) was a Dutch mathematician who made big contributions to hydrodynamics and differential equations — recognised as one of the first mathematical physicists. He was the third generation of the mathematically eminent family of Bernoulli.

In summary, the important class of incompressible and irrotational fluid flows may be described by any velocity potential φ satisfying Laplace's equation $\nabla^2\varphi = 0$: the irrotational velocity field is then $\mathbf{v} = \boldsymbol{\nabla}\varphi$; and the pressure field is given by Bernoulli's equation (1.26). Such velocity and pressure fields satisfy the continuity and Euler equations.

Example 1.35: Consider the flow in a corner $\mathbf{v} = -x\mathbf{i} + y\mathbf{j}$ with velocity potential $\varphi = -\frac{1}{2}x^2 + \frac{1}{2}y^2$. Ignoring gravity, the corresponding pressure field is

$$\frac{p}{\rho} = C - \frac{\partial\varphi}{\partial t} - \frac{1}{2}|\mathbf{v}|^2 = C - \frac{1}{2}(x^2 + y^2)\,.$$

Observe that in this flow the pressure decreases from a maximum at the origin. This is an example of the fairly general *Venturi effect* namely that the pressure decreases with increasing fluid speed. Hence in this corner flow a region of high pressure occurs near the origin because it is there that the flow is the slowest. Bernoulli's equation quantifies the Venturi effect.

The typically high pressure in a corner flow is why cars have intake vents for air in the corner at the bottom of the windscreen. Being a corner flow, albeit not a right-angle corner, the pressure is high in this region and so flow into the vent and thence into the car's cabin is encouraged. Conversely, the vents to exhaust air out of a car's cabin are typically located on the back pillars. At these locations the air moves very quickly backwards, relative to the car, and so there is a region of low pressure there which helps suck the air out of the cabin. Both of these placements aid air circulation inside the car's cabin.

Activity 1.R → Watch the videos "Bernoulli's Equation" [BE91] and "Pressure Fields" [PF] ([CD08], Venturi Effect) and understand the Venturi effect and how it is used in practice. Do the following exercises. Send in to the lecturer for feedback solutions for Exercises 1.36–1.39, 1.40 and 1.41.

1.6.5 Fluid exercises

Ex. 1.36: Prove the identity $\boldsymbol{\nabla}\left(\frac{1}{2}|\mathbf{v}|^2\right) = \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$ for an irrotational vector field $\mathbf{v}$.

Ex. 1.37: Summarise necessary conditions for validity of the Bernoulli equation.

Ex. 1.38: Discuss the steady Bernoulli equation from the point of view of the energy conservation principle.

Ex. 1.39: Restrict attention to incompressible flow for which ρ is constant. Take the curl of Euler's equation (1.17), interchange orders of differentiation, expand the differential operators when stuck, and proceed to derive the *vorticity equation*

$$\frac{D\boldsymbol{\omega}}{Dt} = -(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}\,. \tag{1.27}$$

Hence deduce that if an incompressible fluid flow is ever free of vorticity, $\boldsymbol{\omega} = \mathbf{0}$, then the flow is irrotational for all time. This persistence of irrotational flow shows that it is indeed seen in many situations and is thus an important class of fluid flows.

Ex. 1.40: The velocity potential for a two-dimensional fluid flow is $\varphi = (-3x + 4y)\cos t$.

(a) Is the flow irrotational?

(b) Is the flow divergence free?

(c) If so, determine the pressure field.

Ex. 1.41: Suppose water in a U-tube is oscillating about a state of equilibrium. Just consider the flow in the exactly vertical section of one upright of the U-tube. Assume that the displacement of the surface is $z = \eta(t) = A\cos\omega t$ with the origin of z being the equilibrium level of the water's surface. For definiteness, say $A = 10\,\text{cm}$, the frequency $\omega = 2\pi/T$ with period $T = 4\,\text{sec}$, $g = 980\,\text{cm/sec}^2$ and $\rho = 1\,\text{g/cm}^3$.

(a) Draw a diagram of the physical situation.

(b) What is the velocity field in the upright section of this U-tube?

(c) Deduce a velocity potential φ for the flow in this section.

(d) At the surface of the water, wherever it happens to be, the atmosphere fixes the pressure to be constant in time (and space if necessary), say 0. Determine the $C(t)$ that appears in Bernoulli's equation.

(e) Hence use Bernoulli's equation to determine the pressure 20cm below the instantaneous water surface at $t = 0\,\text{sec}$, $t = 1\,\text{sec}$ and $t = 2\,\text{sec}$.

Ex. 1.42: Consider the incompressible and irrotational flow field of a uniform wind blowing across a circular cylinder of radius 1 that was given earlier, namely

$$\mathbf{v} = \left(1 + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right)\mathbf{i} - \frac{2xy}{(x^2 + y^2)^2}\mathbf{j} = \left(1 + \frac{y^2 - x^2}{r^4}\right)\mathbf{i} - \frac{2xy}{r^4}\mathbf{j},$$

using r to denote $\sqrt{x^2 + y^2}$.

(a) Draw a picture of this velocity field with the contours of the velocity potential superimposed.

(b) Show that the velocity field on the surface of the cylinder $r = 1$ is everywhere tangential to the cylinder and the fluid indeed does not cross the cylinder's surface.

(c) Neglecting gravity, determine the pressure field around the surface of the cylinder $r = 1$ as a function of, say, θ.

That there is apparently no drag on the cylinder is called D'Alembert's paradox — its resolution depends upon a small amount of friction to trigger a fundamental change in the shape of the flow.

(d) Observe that the pressure field is symmetric about both the x-axis and the y-axis. Hence immediately deduce that the net force of the wind on the cylinder is zero!

(e) Now superimpose upon this flow that of the "bathtub vortex", namely add to the velocity field

$$\frac{\Gamma}{2\pi}\left(-\frac{y}{r^2}\mathbf{i}+\frac{x}{r^2}\mathbf{j}\right),$$

The corresponding addition to the velocity potential is $(\Gamma/2\pi)\tan^{-1}(y/x)$.

where Γ is a constant (which in the next section we call the circulation) proportional to the strength of the vortex. This field is approximately that established by spinning the cylinder about its axis.

i. For various Γ draw pictures of the velocity field and observe that the fluid still flows around the cylinder but is now asymmetric.

ii. Determine the velocity field on the surface of the cylinder and again observe that no fluid crosses the cylinder's surface.

iii. Determine the pressure field about the cylinder and again note, by symmetry, that there is no net force in the x-direction.

This "lift" is what makes all sorts of sports balls swerve, float or dip in the air when they have been hit with spin.

iv. However, integrate around the cylinder the component of the pressure in the y-direction to determine that the net force in the y-direction, the lift perpendicular to the wind, is proportional to the circulation Γ.

1.7 Answers to selected exercises

1.3

17. Streamlines are $y = x + \text{const}$ so the pollutant travels along the line $y = x - 1$ to the right and up;

19. Streamlines are $y = \sqrt[3]{3x + C}$ so the pollutant travels along the (reflected) cubic $y = \sqrt[3]{3x - 5}$ to the right and up;

21. Streamlines are simply $y = \text{const}$ so the pollutant travels to the left along $y = 1$.

1.16

1. $\boldsymbol{\nabla}\cdot(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 3 \neq 0$ does not;

3. $\boldsymbol{\nabla}\cdot(e^x \cos y\,\mathbf{i} + e^x \sin y\,\mathbf{j}) = 2x^2 z \neq 0$ does not;

7. $\boldsymbol{\nabla}\cdot(e^x\mathbf{i} + ye^{-x}\mathbf{j} + 2z\sinh x\,\mathbf{k}) = 0$ may do.

1.29

21. $\nabla^2 f = 14$ so f cannot be a velocity potential;

25. $\nabla^2 f = 2xy/z^3$ so f cannot either.

1.30 Only number 5 of the odd problems is irrotational, $\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}$, so consider further only this

5. Using r to denote $\sqrt{x^2+y^2+z^2}$ and noting $\frac{\partial r}{\partial x} = \frac{x}{r}$ etc, compute $\boldsymbol{\nabla}\cdot(\frac{x}{r^3}\mathbf{i} + \frac{y}{r^3}\mathbf{j} + \frac{z}{r^3}\mathbf{k}) = 0$ is incompressible (this is the flow of a point source of fluid) and $\varphi = -1/r$ is the corresponding velocity potential;

1.8 Summary

- Scalar and vector fields are fundamental to the description of fluid dynamics. The field lines of the velocity field are called streamlines and in two-dimensions are determined by solving $\frac{dy}{dx} = \frac{v}{u}$ (§1.1).
- Recall the formula for the gradient in Cartesian coordinates

$$\boldsymbol{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k}\,.$$

  The gradient has many useful properties (§1.2):

  - it points in the direction of maximum increase of φ;
  - it is normal to the level surfaces of φ;
  - it acts on a scalar function and produces a vector field;
  - is used to compute directional derivatives

$$\frac{\partial f}{\partial \mathbf{n}} = \boldsymbol{\nabla} f \cdot \frac{\mathbf{n}}{|\mathbf{n}|}$$

    and the material derivative in fluid flow (§§1.2.1):

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f\,.$$

- Directional derivative acts on a scalar function and produces a scalar which represents the rate of change of this function in a given direction.
- Material derivative can be applied to a scalar or vector function and will produce a scalar or a vector, respectively. It represents the rate of change of a field due to its temporal variation and due to the motion of the observer through the field.
- Recall the formula for the divergence of a vector field $\mathbf{v} = u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$ in Cartesian coordinates (§1.3):

$$\operatorname{div}\mathbf{v} = \boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\,.$$

  The divergence acts on a vector field and produces a scalar. The divergence expresses a fundamental property of the vector field, namely its local "pointing awayness" (§§1.3.1). This is critical to its appearance in the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \mathbf{v}) = 0\,,$$

expressing conservation of mass in a (possibly) compressible fluid. In an incompressible fluid (i.e. constant density) this equation reduces to

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0\,.$$

- Stream function ψ can be introduced for two-dimensional flows so that $u = \frac{\partial \psi}{\partial y}\,, \quad v = -\frac{\partial \psi}{\partial x}$. The physical meaning of stream function for incompressible fluid is the flow rate along the flow pipe bounded by the streamlines. Stream function is constant along the streamlines, i.e. the geometrical meaning of stream function is that its level curves are streamlines.

- Recall the formula for the curl of a vector field in Cartesian coordinates (§1.5):

$$\operatorname{curl}\mathbf{v} = \boldsymbol{\nabla}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}\,.$$

  The curl acts on a vector field and produces a vector field. The curl expresses another fundamental property of a vector field, namely the amount of its local "rotation" (§§1.5.1). In fluid dynamics $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{v}$ is called the vorticity.

- Irrotational vector fields may be written as the gradient of a scalar function. In irrotational fluid flow, $\mathbf{v} = \boldsymbol{\nabla}\varphi$, where φ is called the velocity potential (§§1.2.2). By the continuity equation, the velocity potential of the irrotational flow of an incompressible fluid must satisfy Laplace's equation (§§1.5.2):

$$\nabla^2\varphi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi = 0\,.$$

- Fluid flow must satisfy Euler's equation (§§1.6.1):

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \mathbf{g}\,.$$

  Incompressible, inviscid, isothermal, irrotational flow, with velocity potential φ, is a solution of this equation provided the pressure field is determined from the Bernoulli's equation (§§1.6.4):

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\rho} - \mathbf{r}\cdot\mathbf{g} = C(t)\,.$$

- Small amplitude waves on the water surface are described by the wave equation

$$\frac{\partial^2 \eta}{\partial t^2} = c^2\frac{\partial^2 \eta}{\partial x^2}\,,$$

  where $c = \sqrt{gh}$ is the wave propagation speed.

Module 2 Vector Integration and Applications

In the previous module we investigated the main differential operations in space and how they are involved in fluid flow. It will be no surprise to learn in this module that integration in space is also important in numerous applications. Integration is essentially a sum over some domain to determine some net quantity. The critical integration principle is that of equal weight for equal length, area, or volume. Observe this principle in the definitions.

As in Module 1, we investigate the role that these integrals and their properties have in fluid dynamics. Remember though, they have a very wide range of applications; we only use fluid dynamics as just one example area of application. The mathematics developed herein is used in disciplines such as electromagnetism, gravity, quantum physics, biology, etc.

Module contents

As for Module 1, some material usefully supporting the concepts developed in this module is to be found on the World Wide Web (WWW). For example, at the time of writing the Connected Curriculum Project has material which you may like to explore further at our mirror site at `http://www.math.montana.edu/frankw/ccp/multiworld/topic.htm`.

2.1 Circulation is a work integral

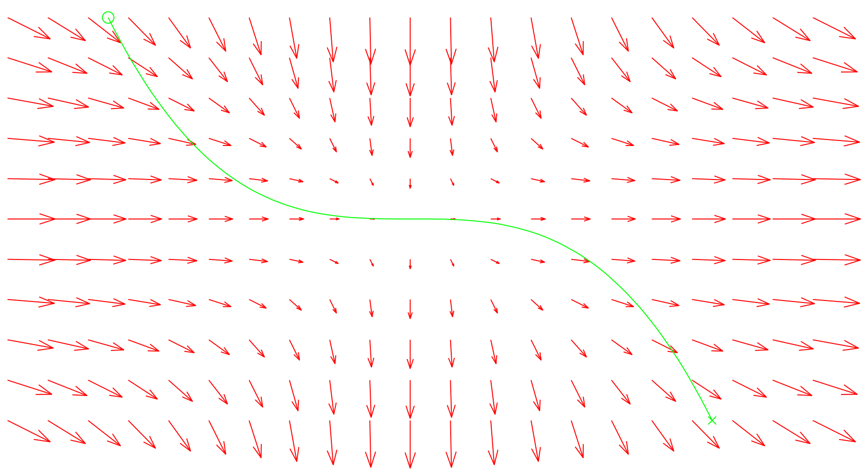
The most familiar integral is the integral in one variable, for example, $\int f(x)\,dx$. This integral may be thought of as being taken along the x-axis. However, in applications to three-dimensional space we often need to integrate along any given path in that space. These are termed *line integrals*. Depending upon the application they effectively sum some property along the path. For example, the length of a curving support cable is the sum (hence an integral) of all the small nearly straight segments it could be "cut" into. There are two sorts of line integrals: integrals of scalar functions, and integrals of vector functions. They share many of the properties of the ordinary integral and have many uses.

Objectives:

- recall the properties and how to compute the line integral of a vector function;
- similarly recall the line integral of a scalar function;
- introduce the circulation as a line integral of the velocity field of a fluid flow.

Reading 2.A → Study §10.1 in Kreyszig [Kre06, pp. 420–425], but take no notice of his subsection entitled "Other forms of line integrals" [Kre06, p. 424].

Example 2.1: In the following picture a vector field is plotted, call it $\mathbf{v}$. Superimposed is a path of integration C going from $\circ$ to $\times$. In going from $\circ$ to $\times$ along the path, see that the vector field, although at an angle to the curve, always points more-or-less in the direction of travel along C. Thus $\mathbf{v}\cdot d\mathbf{r}$ is always positive (or at least zero), and hence we deduce the work integral $\int_C \mathbf{v}\cdot d\mathbf{r} > 0$. If the direction of integration was reversed, from $\times$ to $\circ$, then $\mathbf{v}$ would more-or-less oppose the direction of travel and the work integral would be negative.

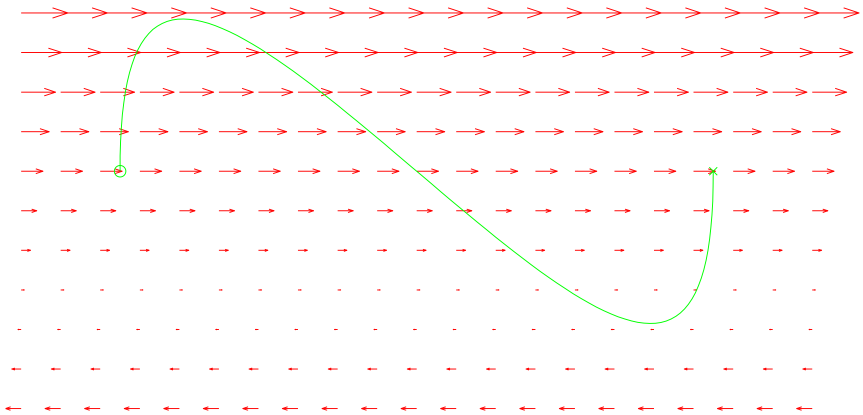


2.1.1 Work integral exercises

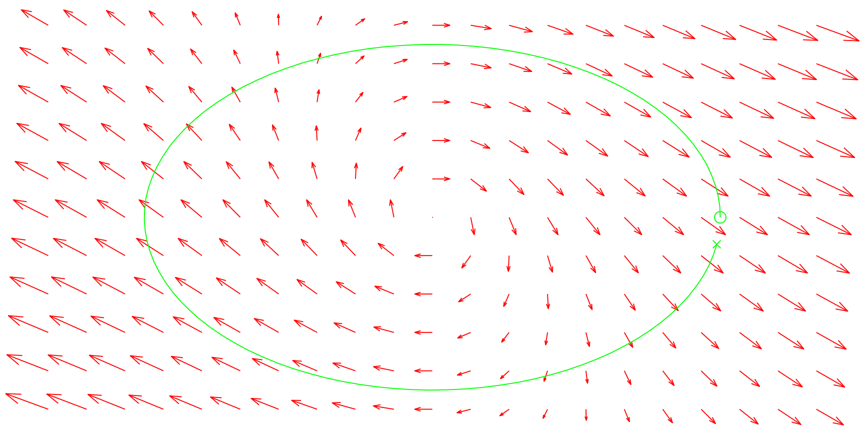
Activity 2.B → Do a selection of Problems 1–12 & 14 from Problem Set 10.1 [Kre06, pp. 425]. In our experience, most people find parameterisation of curves one of the most difficult aspects — practise it. Send in to the lecturer for feedback solutions for Problems 5 and 7 from [Kre06, p. 425] and the first two parts of **Exercise 2.2** below.

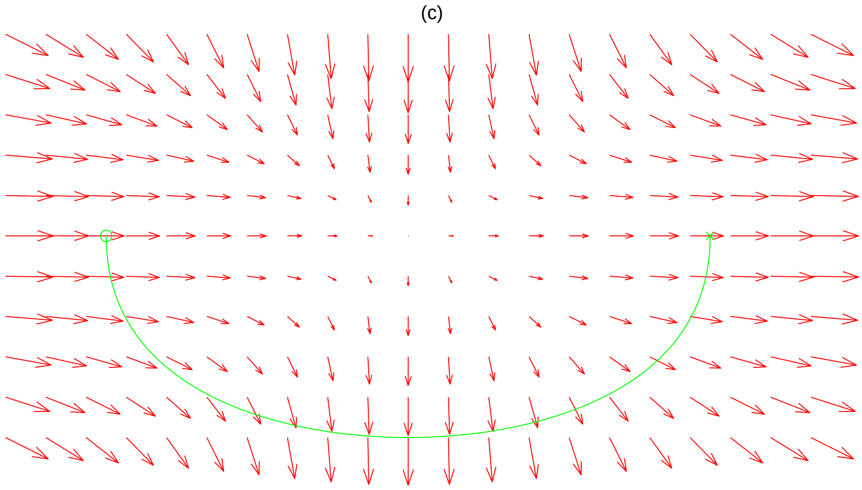
Exercise 2.2: For the following vector fields and the curves plotted on them, going from ∘ to ×, estimate whether the "work integrals" along the curve would be positive, negative, or approximately zero.

(a)



(b)





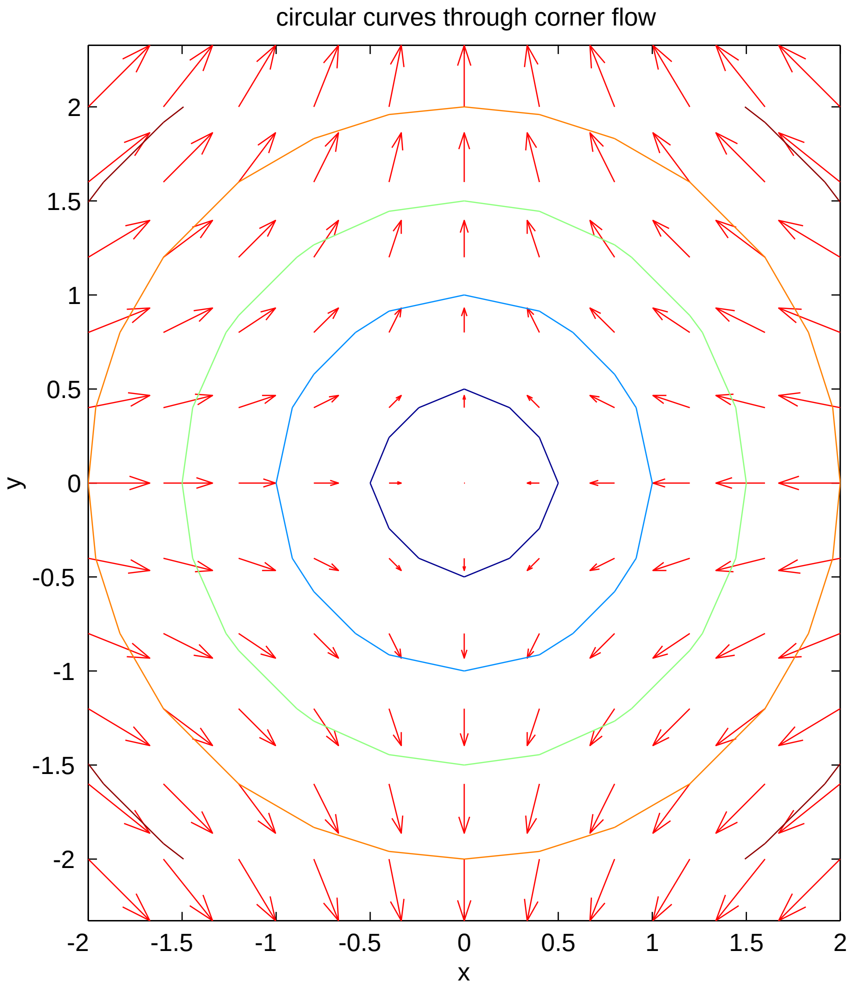
Perhaps investigate some CCP material at http://www.math.montana.edu/frankw/ccp/multiworld/traveling/line-integrals/learn.htm.

2.1.2 Circulation lifts aeroplanes

In the flow of a fluid with a velocity field $\mathbf{v}$, the *circulation* along any *closed* curve C is defined to be the line integral of the velocity field about the closed curve C:

$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{r} \,. \tag{2.1}$$

Example 2.3: Consider the circulation in the corner flow $\mathbf{v} = -x\mathbf{i} + y\mathbf{j}$ around circular paths centred on the origin as shown below:




```
[x,y]=meshgrid(-2:0.4:2);
quiver(x,y,-x,y,'w')
hold on
contour(x,y,sqrt(x.^2+y.^2))
hold off
```

- A circle of radius R may be parameterised by $\mathbf{r} = R\cos t\,\mathbf{i} + R\sin t\,\mathbf{j}$ for $0 \le t \le 2\pi$ (integrating in the default direction of anti-clockwise).
- The velocity field evaluated on C is $\mathbf{v} = -R\cos t\,\mathbf{i} + R\sin t\,\mathbf{j}$.

- Thus

$$\begin{aligned}
\Gamma &= \oint_C \mathbf{v}\cdot d\mathbf{r} \\
&= \int_0^{2\pi} (-R\cos t\,\mathbf{i} + R\sin t\,\mathbf{j})\cdot(-R\sin t\,\mathbf{i} + R\cos t\,\mathbf{j})\,dt \\
&= \int_0^{2\pi} 2R^2 \sin t\,\cos t\,dt \\
&= \int_0^{2\pi} R^2 \sin 2t\,dt \\
&= 0\,.
\end{aligned}$$

The circulation is zero around all circles centred upon the origin. Actually, the circulation is zero around *all* closed curves in this velocity field.

Example 2.4: However, consider the circulation around an arbitrary circle in the solid body rotation $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$.

- A circle of radius R centred upon the point (a, b) may be parameterised by $\mathbf{r} = (a + R\cos t)\mathbf{i} + (b + R\sin t)\mathbf{j}$ for $0 \le t \le 2\pi$ (anti-clockwise).
- The velocity field evaluated on C is $\mathbf{v} = -(b + R\sin t)\mathbf{i} + (a + R\cos t)\mathbf{j}$.
- Thus

$$\begin{aligned}
\Gamma &= \int_0^{2\pi} [-(b + R\sin t)\mathbf{i} + (a + R\cos t)\mathbf{j}]\cdot[-R\sin t\,\mathbf{i} + R\cos t\,\mathbf{j}]\,dt \\
&= \int_0^{2\pi} bR\sin t + R^2\sin^2 t + aR\cos t + R^2\cos^2 t\,dt \\
&= \int_0^{2\pi} bR\sin t + aR\cos t + R^2\,dt \\
&= 2\pi R^2\,.
\end{aligned}$$

Observe the circulation is just twice the area of the disk enclosed by the circular curve, independent of the centre of the circular path.

Note that the circulation is zero for the irrotational corner flow, and twice the area enclosed by the curve in the solid body rotation. These examples suggest that the circulation has something to do with the vorticity of the fluid flow, $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{v}$, as the vorticity is zero in the first and the constant two in the second. This is indeed the case as is proved later by Stokes' theorem (§2.5): the circulation around a curve is exactly the same as the integral of the vorticity over a surface enclosed by the curve. Hence in irrotational flow the circulation is always zero as the vorticity is zero by definition.

Exercise 2.5: Show that the circulation around all circles centred upon the origin in the bathtub vortex flow $\mathbf{v} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)$ is constant, independent of the radius of the circular path.

But earlier we found that the bathtub vortex was an irrotational flow. How is it that the circulation is non-zero? The resolution of these facts is that the vorticity, and hence the circulation, is concentrated entirely at the singular point $x = y = 0$ where the flow velocity "goes to infinity" (it is non-differentiable). Strictly speaking, the flow is irrotational everywhere except at $x = y = 0$. Thus there may be a contribution to the circulation from the origin, and in this case there is.

Circulation is intimately involved in the lift of an aeroplane's wing, or a sailing boat's sail. One view is that a wing generates lift because the faster airflow above the wing, compared to that below, has lower pressure by the Venturi effect, and hence generates a net lifting force. Consider computing the circulation in the air flow around a wing when the wind is blowing left to right (or equivalently when the wing moves to the left). Take the path of integration to be anticlockwise in the air adjacent to the wing. Then along the top part of the wing the air moves fast and thus there is a large negative contribution to the circulation; along the bottom half of the wing the air moves more slowly and so there is a smaller positive contribution to the circulation. Thus there is a net circulation (negative) around the wing. This non-zero circulation is a direct consequence of the differential air speed above and below the wing and is thus proportional to the lift of the wing.

Exercise 2.6: Compute the circulation around the circular cylinder, C is anticlockwise around $r = 1$, in the flow of Problem 1.42 of §§1.6.5 in the previous module. Show the circulation is precisely Γ and is thus proportional to the lift.

The circulation tied up with a wing is as if there was a vortex (like the bathtub vortex) within the wing. Indeed, the first good theory of wings is based precisely upon this idea. Prandtl's lifting line theory replaces a wing by a vortex with the same circulation. This theory could show, for example, that drag is minimised with a tapered wing—an elliptical planform being the strict optimum.

This picture of the vorticity "bound" within a wing has some far reaching consequences. Recall the basic vector identity is that $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) = 0$, and hence the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{v}$ is divergence free. A consequence is that the field lines of vorticity, the vortices themselves, cannot appear or disappear, they either form a closed loop, as in a smoke ring, or stretch to infinity, as in the circular cylinder where the vortex lies on the infinite z-axis. Thus the bound vortex within a wing has to go somewhere when it exits from the wing tip! The bound vorticity turns a right-angle at the wing-tip and trails back behind the wing. This trailing wing-tip vortex can often be seen when a plane lands or takes off in humid conditions. The vorticity is intense, and the rotational flow in the trailing vortex may be so rapid that the pressure is lowered within the core of the vortex (the Venturi effect again) so that water vapour condenses and becomes visible. In principle, the trailing vortex lines stretch from the wing-tips all the way back to the airport that the plane took off from; back at the airport the two trailing vortices rejoin to form a closed loop. Such vorticity only decays slowly through internal friction and through being disrupted by turbulence. These immensely strong vortices at airports, strong enough to completely flip a small plane, place a severe constraint upon the minimum distance between planes that use a runway to take off and land. An understanding of the phenomena depends upon the properties of vorticity and circulation.[1]

2.1.3 Recall line integrals of scalar functions

You might find Kreyszig's subsection on "Other forms of line integrals" [Kre06, pp. 424] somewhat misleading. The critical issue is to sensibly define the integral of a scalar function f along a curve C in space. Using the principle that equal lengths of parts of C have equal importance in the integral, define

$$\int_C f(\mathbf{r})\,ds = \lim_{N\to\infty} \sum_{n=1}^{N} f(\mathbf{r}_n)\Delta s_n\,, \tag{2.2}$$

where we cut the curve C into N small pieces of length Δs_n and $\mathbf{r}_n$ is a point of C in the nth piece. This definition implies equal parts of C have equal weight in the integral.

To obtain a practical formula for the computation of such a line integral we convert it into an ordinary integral. Describe the curve C by *any* parameterisation, that is, C is the set of points $\mathbf{r}(t)$ for $a \le t \le b$. Then, cutting the t interval $[a, b]$ into N pieces of length Δt_n corresponds to cutting C into N pieces of length

$$\Delta s_n = \frac{ds}{dt}\Delta t_n = \left|\frac{d\mathbf{r}}{dt}\right| \Delta t_n\,.$$

Hence the expression for the sum over C becomes

$$\sum_{n=1}^{N} f(\mathbf{r}_n)\Delta s_n = \sum_{n=1}^{N} f(\mathbf{r}(t_n)) \left|\frac{d\mathbf{r}}{dt}\right| \Delta t_n \approx \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| dt\,.$$

Thus we evaluate line integrals of scalar functions via

$$\int_C f(\mathbf{r})\,ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| dt\,. \tag{2.3}$$

Example 2.7: Find the line integral of $f = (x^2 + y^2 + z^2)^2$ over the helix in Example 2 of Kreyszig.

[1] For interest peruse the recent article by Phillippe Spalart "Aeroplane trailing vortices" in *Annual Review of Fluid Mechanics*, **30**, pp.107–138, 1998, which reviews the formation, motion and persistence of trailing vortices relevant to the safety and productivity of air travel.

Solution: The first task is always to parameterise the curve C, here from Kreyszig [Kre06, p. 422] $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 3t\mathbf{k}$ for $0 \leq t \leq 2\pi$. Then

$$\left|\frac{d\mathbf{r}}{dt}\right| = |-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 3\mathbf{k}| = \sqrt{10}\,.$$

On the curve C, the function f is evaluated to be

$$f = (x^2+y^2+z^2)^2 = \left(1+9t^2\right)^2 .$$

Thus we deduce

$$\begin{aligned}\int_C f\,ds &= \int_0^{2\pi} \left(1+9t^2\right)^2 \sqrt{10}\,dt \\ &= \sqrt{10}\left[2\pi + 6(2\pi)^3 + \frac{81}{5}(2\pi)^5\right] \\ &= 506,391.93\,.\end{aligned}$$

Example 2.8: How long is the arc $\mathbf{r} = t\mathbf{i}+t^2\mathbf{j}+\frac{2}{3}t^3\mathbf{k}$ connecting $A(0,0,0)$ to $B(1,1,2/3)$?

Solution: The first fact to recall is that the length of a curve C is simply $\int_C 1\,ds$. Then, since $0 \leq t \leq 1$ parameterises the curve from A to B,

$$\begin{aligned}\text{length} &= \int_C 1\,ds \\ &= \int_0^1 \left|\frac{d\mathbf{r}}{dt}\right| dt \qquad \text{by evaluation formula} \\ &= \int_0^1 \sqrt{1+4t^2+4t^4}\,dt \qquad \text{as } \mathbf{r}' = \mathbf{i}+2t\mathbf{j}+2t^2\mathbf{k} \\ &= \int_0^1 1+2t^2\,dt \qquad \text{upon simplifying} \\ &= \frac{5}{3}\,.\end{aligned}$$

2.1.4 Scalar integral exercises

Activity 2.C → Do a selection of Problems 15–18 from Problem Set 10.1 [Kre06, p. 426]. Send in to the lecturer for feedback solutions for Problems 15 and 17 from [Kre06, p. 426].

2.2 Scalar potentials lead to path independence

In the previous section you are reminded that in general a line integral depends upon the details of the path joining the two end points. See Theorem 2 in §10.1 [Kre06, p. 425]. However, some important integrals are independent of the details of the path. For example, the circulation in irrotational flow is independent of path.

Objectives:

- to show that line integrals of vector functions that have a scalar potential are independent of path;
- to be able to evaluate path independent integrals by finding the scalar potential of a suitable vector function.

Reading 2.D → Study Section 10.2 in Kreyszig [Kre06, pp. 426–431].

Activity 2.E → Do a selection of Problems 1–20 from Problem Set 10.2 [Kre06, p. 432]. Also translate these problems phrased in terms of differential forms into equivalent problems phrased in terms of a vector field. Send in to the lecturer for feedback solutions for Problems 3 and 11 from [Kre06, p. 432].

For a discussion of work, potentials and line integrals perhaps look at http://www.math.montana.edu/frankw/ccp/multiworld/traveling/work-energy/learn.htm.

Example 2.9: Consider Example 3 in §10.2 [Kre06, p. 430] and write it in vector terms. The integral

$$\begin{aligned} I &= \int_C \left[2xyz^2\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz\right] \\ &= \int_C \mathbf{F}\cdot d\mathbf{r}\,, \end{aligned}$$

is a line integral over the curve C of the vector field

$$\mathbf{F} = 2xyz^2\mathbf{i} + (x^2z^2 + z\cos yz)\mathbf{j} + (2x^2yz + y\cos yz)\mathbf{k}\,.$$

Following the working given in Kreyszig, deduce that $\mathbf{F} = \boldsymbol{\nabla} f$ where the scalar potential is $f = x^2yz^2 + \cos yz$. Hence the integral is as given.

2.2.1 A comment on irrotational flow

Note that the velocity field of irrotational flow satisfies, by definition, $\boldsymbol{\nabla}\times\mathbf{v} = \mathbf{0}$. Thus irrotational flow always possesses a velocity potential ϕ such that $\mathbf{v} = \boldsymbol{\nabla}\phi$. Hence the circulation computed around any path in an irrotational flow is always zero.

Always? How then did we compute the circulation around the cylinder in Exercise 1.42 of §§1.6.5 to be non-zero? The answer is that in this case the region of flow is not simply connected. The cylinder makes a hole in the domain of the velocity field. Thus the circulation computed with any path that goes around the cylinder *may* be non-zero; in this case the circulation is Γ. However, in this irrotational flow is is indeed true that the circulation must be zero for any closed curve that does *not* go around the cylinder.

Similarly the two-dimensional domain of air flow around a wing's cross-section is also *not* simply connected. The wing forms an "island" in the air flow. Thus, although in irrotational flow the wing can generate circulation, which generates faster flow above the wing and slower flow below, which in turn generates lift to keep the aeroplane in flight.

2.3 Surface integrals measure flux

The previous parts of this module define an integral over a curve in space. Essentially a line integral is a sum over all its parts. But for other purposes we also need to similarly sum over curved *surfaces*. This leads to a natural generalisation of integration. The generalisation is in two ways: from one-dimensional curves (from the previous sections) to two-dimensional surfaces; and from double integrals in the plane (from Algebra & Calculus II) to two-dimensional surface integrals.

One application where we have to be very good at describing surfaces is in ship hydrodynamics and water waves. Whether we wish to compute the wave drag of a ship or the breaking of a wave, a fundamental difficulty is that the surface of the water keeps moving. To keep track of the surface and determine how its shape affects the water flow we thus need completely general techniques and formulae for the surface. Such mathematical tools are developed in this section.

Objectives:

- to recall the properties and how to compute double integrals;
- to revise how to describe surfaces in space and determine normals and tangent planes;
- to understand and compute integrals over curved surfaces;
- understand and compute surface flux integrals.

2.3.1 Revise double integrals

The first task is to revise double integrals. These are integrals over a region in a plane. They may be used to compute area of regions, volumes of solids, and the centre of gravity and moments of inertia of two-dimensional objects. Being over a flat planar region the basic concepts of the double integral are the same as a surface integral, our subsequent task is to generalise the treatment to a curved surface.

Activity 2.F → Revise double integrals as in §10.3 in Kreyszig [Kre06, pp. 433–438]. Ensure that you can do the sorts of problems in Problem Set 10.3 [Kre06, pp. 438–439]. Send in to the lecturer for feedback solutions for Q3, 7 and 11 from [Kre06, p. 438]. Peruse Green's Theorem in §10.4 [Kre06, pp. 439–444].

Perhaps look at CCP material at `http://www.math.montana.edu/frankwccp/multiworld/multipleIVP/volume/learn.htm`.

2.3.2 The description of surfaces in space

But before we can define and compute integrals over curved surfaces, we need to know how to mathematically describe such surfaces. Only then will we know how to start the algebraic machinations. *In these sections it is essential that you draw diagrams and graphs to help visualise the problem in hand—marks in assignments and the examination will be allocated to your pictures.*

Fundamental properties of a surface are its *tangent vectors*, *tangent plane*, and the *normal vector* & *unit normal* vector to the surface.

Reading 2.G → Study §10.5 in Kreyszig [Kre06, pp. 445–448].

Note that Kreyszig uses u and v to parameterise surfaces in space. This is fine, but in any application to fluid dynamics it may cause confusion with the components of the fluid velocity, normally $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. In fluid dynamics we typically use other names for the parameters of a surface.

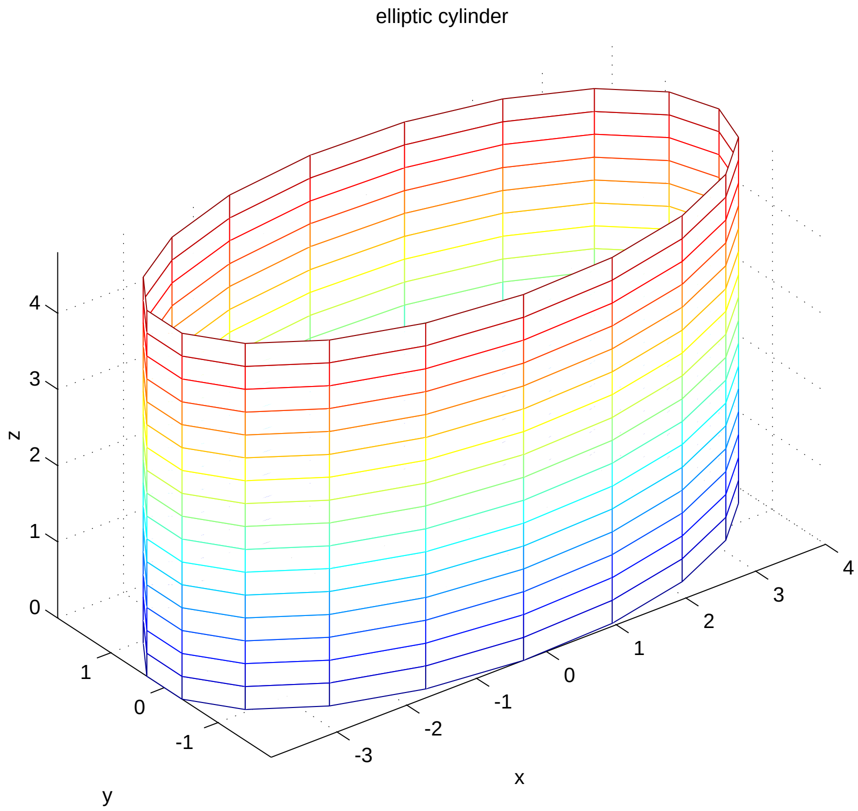
Activity 2.H → Do problems from Problem Set 10.5 [Kre06, pp. 448–449]. Students usually find the parameterisation of surfaces to be the most difficult aspect — practise it. Send in to the lecturer for feedback solutions for Q4, 6, 18 and 19 from [Kre06, pp. 448–449].

Example 2.10: MATLAB may be used to help draw pictures of surfaces. Consider the elliptic cylinder of Problem 3 in §10.5 [Kre06, p. 448] with $a = 4$ and $b = 2$ parameterised by

$$\mathbf{r} = 4\cos v\,\mathbf{i} + 2\sin v\,\mathbf{j} + u\mathbf{k}\,.$$

Or `surf` could be used instead of `mesh`.

This surface may be drawn by the following.




```
[v,u]=meshgrid( ...
linspace(0,2*pi,20),0:.3:5);
x=4*cos(v);
y=2*sin(v);
z=v;
mesh(x,y,z)
axis('equal')
```

The range for v of $[0, 2\pi]$ is chosen to go once around the cylinder; the range for u is rather arbitrary here, but in general you would experiment a little and choose something suitable. The *parameter curves* are visible in the picture: $v = \text{const}$ (and thus u variable) are the straight lines parallel to the z-axis; whereas the parameter curves $u = \text{const}$ (and thus v variable) are ellipses at constant z. A normal vector may be computed via

$$\begin{aligned} \mathbf{N} &= \mathbf{r}_v \times \mathbf{r}_u \\ &= (-4\sin v\,\mathbf{i} + 2\cos v\,\mathbf{j}) \times \mathbf{k} \\ &= 2\cos v\,\mathbf{i} + 4\sin v\,\mathbf{j}\,. \end{aligned}$$

A unit normal is obtained simply by dividing by its length, here

$$\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N} = \frac{1}{\sqrt{1 + 3\sin^2 v}}(\cos v\,\mathbf{i} + 2\sin v\,\mathbf{j})\,.$$

Of course the negative of the above is the other possibility for a unit normal—it depends whether you need $\mathbf{n}$ to point into or out of the cylinder.

Note that other textbooks and other courses may use the notation that *any* normal vector is denoted $\mathbf{n}$, whereas a *unit* vector is indicated by a "hat" as in $\hat{\mathbf{n}}$ (the hat may be used for any unit length vector). For example, instead of Equation (2) in §10.6 [Kre06, p. 449] others may write that S has a normal vector

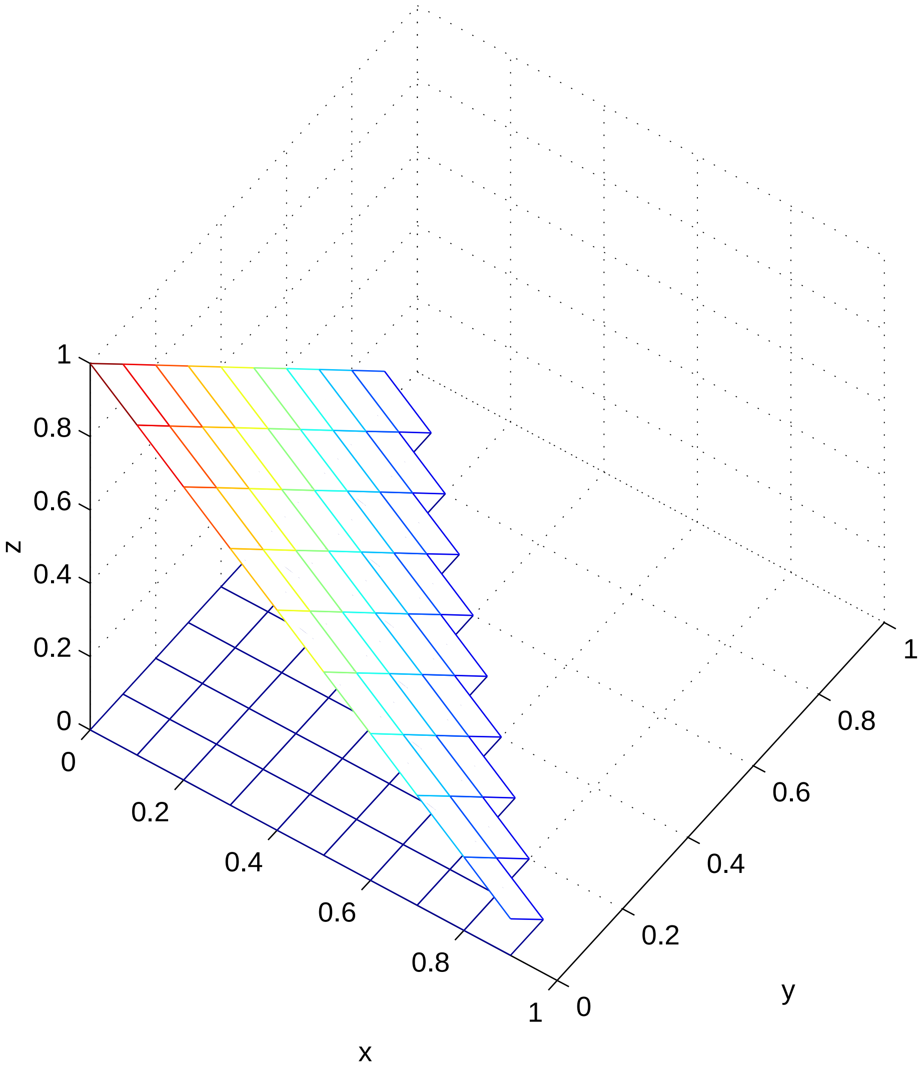
$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v \quad \text{and unit normal vector} \quad \hat{\mathbf{n}} = \pm\frac{1}{|\mathbf{n}|}\mathbf{n}\,.$$

2.3.3 Surface integrals

We investigate the basics of two types of integrals over a surface: surface integrals of scalar functions; and surface integrals of vector functions. The first is generally used to determine properties of the surface itself: its area, its centre of mass, the total energy in the surface, etc. The second is typically used to determine processes on or through a surface: the flux of fluid matter across the surface; the flux of momentum carried through, etc.

Surface integrals are essentially a sum of some property, as expressed by the integrand, over some curved surface, the domain of integration. For example, the total amount of contaminant, such as oil, on the surface of the sea is the integral of the density of the oil over the undulating, even splashing, water's surface.

The basic difficulty is that a curved surface generally slopes and so areas on the surface are magnified by their slope when compared to areas of a convenient reference plane. For a simple example, consider the plane surface $z = 1 - x - y$ drawn below.




```
[x,y]=meshgrid(0:.1:1);
junk=find(x+y>1);
x(junk)=repmat(nan,size(junk));
y(junk)=x(junk);
z=1-x-y;
mesh(x,y,z)
view(35,50)
hold on
mesh(x,y,zeros(size(x)))
hold off
axis('equal')
```

Each little patch of this sloping surface is $\sqrt{3}$ times as big as the corresponding patch in the xy-plane below. Thus, for example, since the area of the triangle in the xy-plane is $1/2$, the area of the sloping plane is $\sqrt{3}/2$. Exactly the same considerations apply to curved surfaces, but the considerations are more complicated because the slope of the surface varies across the surface. Furthermore, how did we know that the factor of $\sqrt{3}$

is correct? We delve into these matters in this subsection.

The fundamental principle of a surface integral is that equal sized areas of a surface S should have equal weight in the summation. Thus we define the *surface integral* of a scalar function G as follows. Cut up the surface into N small patches of area ΔA_n. Let a typical value of G on the nth patch be G_n. Then (compare the right-hand side with that for line integrals (2.2) and volume integrals (2.7))

Kreyszig uses a double integral sign $\iint$ whereas others prefer the same number of $\int$'s as there are differentials, here the one dA.

$$\int_S G\,dA \quad \text{or} \quad \iint_S G\,dA = \lim_{N\to\infty} \sum_{n=1}^{N} G_n\,\Delta A_n\,. \tag{2.4}$$

As argued in Kreyszig [Kre06, p. 535], if S is parameterised by $\mathbf{r}(u,v)$, the area ΔA of a small patch of surface Δu by Δv is nearly $|\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v$. Then the above

$$\sum_{n=1}^{N} G_n\,\Delta A_n \approx \sum_{n=1}^{\infty} G_n\,|\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u_n\,\Delta v_n \approx \iint_R G[\mathbf{r}(u,v)]\,|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv\,,$$

by definition of a double integral. Thus surface integrals are *evaluated* by the double integral

$$\int_S G\,dA = \iint_R G[\mathbf{r}(u,v)]\,|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv\,, \tag{2.5}$$

where R is the region of the parameter plane corresponding to the surface S. This formula is essentially Equation (6) in [Kre06, p. 454].

The above formula very neatly reduces to the one derived in Algebra & Calculus II for the transformation of integration variables in a double integral. Consider the special case where the surface S is in the plane $z = 0$ so that its parameterisation is $\mathbf{r} = x(u,v)\mathbf{i} + y(u,v)\mathbf{j}$ for some domain R of the uv-plane. Then

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} \\ &= \mathbf{k}\,(x_u y_v - x_v y_u) \\ &= \mathbf{k}(\text{Jacobian})\,. \end{aligned}$$

Thus (2.5) reduces to

$$\int_S G\,dA = \iint_R G\,[x(u,v), y(u,v)]\,\left(x_u y_v - x_v y_u\right)\,du\,dv\,,$$

as previously seen for double integrals.

However, Kreyszig first considers [Kre06, pp. 449–453] the surface integral of a vector function and argues that it should be denoted and computed by

$$\int_S \mathbf{F}\cdot d\mathbf{A} \quad \text{or} \quad \iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iint_R \mathbf{F}[\mathbf{r}(u,v)]\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv\,. \tag{2.6}$$

That is, this is the integral of the normal component of the vector function $\mathbf{F}$ over the surface S. The differential $d\mathbf{A} = \mathbf{n}\,dA$ is often termed the differential *vector area*. The reason is that $d\mathbf{A}$ denotes both a combination of the magnitude of an element of area, dA, and the orientation of that element of the area, encoded by the unit normal $\mathbf{n}$.

Reading 2.I → Study §10.6 in Kreyszig [Kre06, pp. 449–456].

Take note of the interpretation of $\int_S \mathbf{F}\cdot d\mathbf{A}$ as the *flux* of something across S. For example, if ρ is the density of a fluid flowing with velocity field $\mathbf{v}$, then $\int_S(\rho\mathbf{v})\cdot d\mathbf{A}$ denotes the total flux of fluid material (mass per unit time) across the surface S. Similarly, since ρu is the density of momentum in the x-direction, and $\mathbf{v}$ is the velocity with which it is being carried, then $\int_S(\rho u\mathbf{v})\cdot d\mathbf{A}$ denotes the rate at which x-momentum is being carried across the surface S. We use these fluxes in the next section to derive the continuity and Euler equations.

Activity 2.J → Do a representative range from Problem Set 10.6 [Kre06, pp. 456–458]. Problems 28–30 are somewhat harder and may be treated as extension exercises. Send in to the lecturer for feedback solutions for Q 1, 3, 5, 15 and 17 from [Kre06, p. 456].

2.4 Gauss' divergence theorem transforms volume integrals

Integrals over a three-dimensional volume in space may be evaluated as a triple integral just as the integral over a two-dimensional area in the plane may be evaluated by a double integral. Such volume integrals have routine applications such as determining the total mass or total kinetic energy contained in any given volume. One quantity of immense importance in the debate on global warming is the heat budget for the oceans of the world. The oceans form an immense reservoir of coolness in the global dynamics and its total heat content is proportional to the volume integral of its temperature.

Remarkably, using Gauss'[2] divergence theorem we can transform some volume integrals into integrals over the surface of the volume, and vice-versa. For example, you are asked later to show that the total volume of some 3-D region of space T, computed as $\int_T 1\,dV$ may be also computed by $\frac{1}{3}\int_S \mathbf{r}\cdot d\mathbf{A}$ where S (or $S(T)$) is the surface enclosing T. Thus this theorem allows us to relate processes defined on a surface to associated processes in the volume enclosed by the surface, and vice-versa. This property is used herein to derive the dynamic continuity and Euler's equations of fluid mechanics.

Objectives:

- introduce volume integrals, otherwise known as triple integrals;

[2] Carl Friedrich Gauss (1777–1855) contributed to a wide variety of fields in both mathematics and physics including number theory, analysis, differential geometry, geodesy, magnetism, astronomy and optics. His work has had an immense influence.

- show how and why Gauss' divergence theorem transforms some surface integrals into volume integrals, and vice-versa;
- use Gauss' divergence theorem in conjunction with conservation principles to derive differential equations governing the dynamics of a continuum.

Reading 2.K → Study §10.7 in Kreyszig [Kre06, pp. 458–462].

The first task was to define an integral over a volume T in such a way as to give equal weight to equal sub-volumes. Define the *volume integral* or *triple integral* by (compare the right-hand side with that for line integrals (2.2) and surface integrals (2.4))

$$\int_T f\,dV \quad \text{or} \quad \iiint_T f\,dV = \lim_{N\to\infty} \sum_{k=1}^{N} f_k\,\Delta V_k\,, \tag{2.7}$$

where T is divided into N small sub-volumes, each of volume ΔV_k, with f_k being a representative value of f in the kth sub-volume. Then, for example, if the volume is described by $g(x,y) \le z \le h(x,y)$ for $p(x) \le y \le q(x)$ for $a \le x \le b$, then it may be evaluated as three single variable integrals

$$\int_T f\,dV = \int_a^b \left\{ \int_{p(x)}^{q(x)} \left[\int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz \right] dy \right\} dx\,. \tag{2.8}$$

As we see soon, there are many useful applications of Gauss' *divergence theorem*, that

$$\int_T \nabla\cdot\mathbf{F}\,dV = \int_{S(T)} \mathbf{F}\cdot d\mathbf{A}\,, \tag{2.9}$$

where $S(T)$ is the surface that *encloses* the volume T.

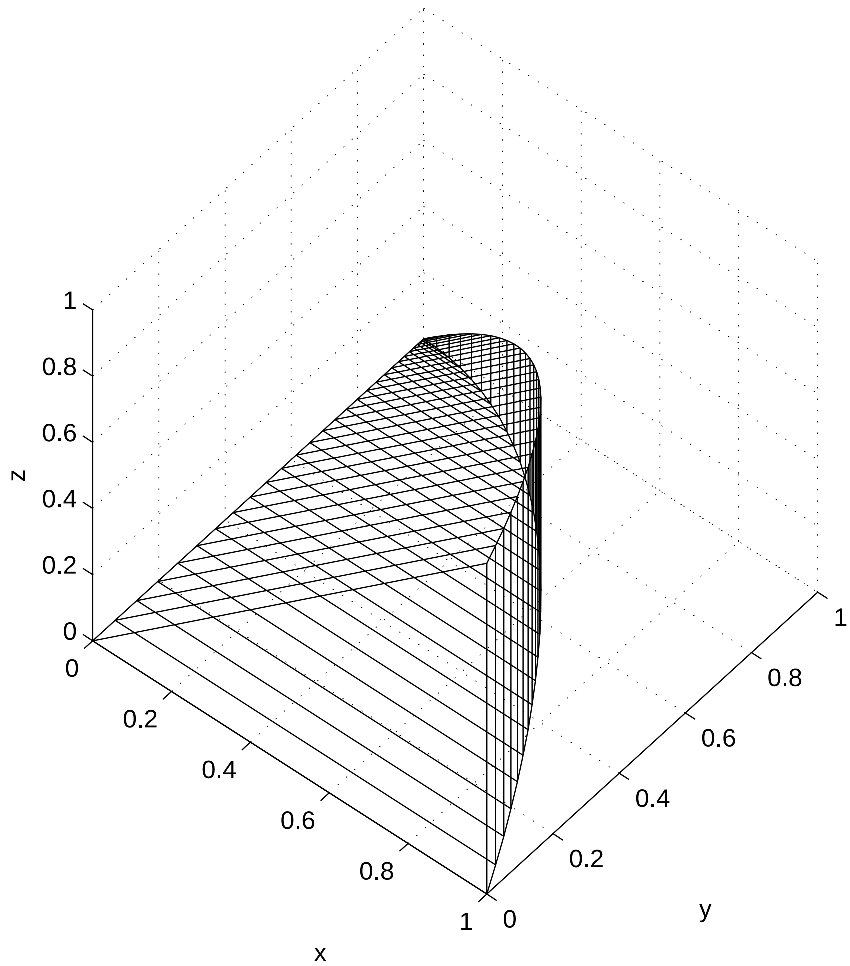
Example 2.11: Find the mass in the region T in the first octant bounded by $y = 1 - x^2$ and $z = x$ when the density of the material is $\rho = 4z$.

Solution: The first task is to visualise the volume T so that we can determine the limits of integration of the volume integral.

The most common error is to get the limits of integration wrong. Draw graphs. Practise.

- First octant means that for the integration reagion $x \ge 0$, $y \ge 0$, $z \ge 0$. Its boundaries are coordinate planes $x = 0$, $y = 0$ and $z = 0$.
- The surface equation $y = 1 - x^2$ is independent of z, therefore z can take any values. This surface is a parabolic cylinder which extends vertically from $z = -\infty$ to $z = \infty$, but we are only interested in its part located in the first octant.
- $z = x$ is a plane which contains the line $x = z = 0$, $-\infty < y < \infty$ and cuts the first actant diagonally.

The integration region lies between these surfaces as plotted below.




```
[x,y]=meshgrid(0:.05:1);
junk=find(y>1-x.^2);
x(junk)=nan*x(junk));
y(junk)=nan*y(junk));
z=x;
mesh(x,y,z)
view(40,50)
hold on
mesh(x,y,zeros(size(x)))
hold off
```

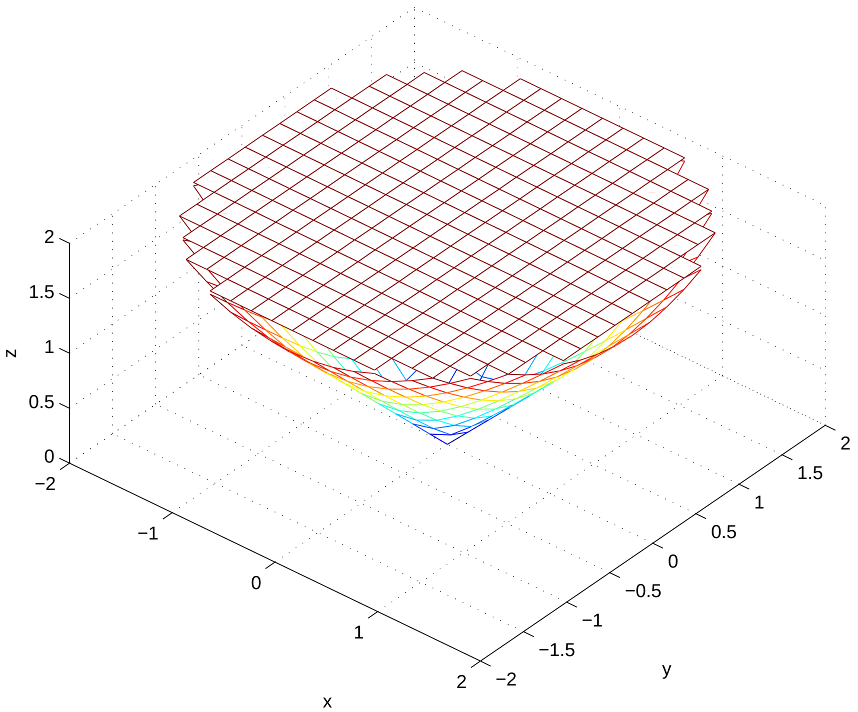
The next task is to choose the order of integration. While the final result is independent of the chosen order the algebraic complexity of calculations may depend on it. The procedure is usually simplified if one avoids "exiting" and "re-entering" the integration region along the chosen integration direction. However this is not a problem here: the integration region is convex, i.e. any two points within the region can be connected by a segment of a straight line which is fully contained within the region. Therefore the integration order can be chosen arbitrarily in this case, for example, $dzdydx$. To choose the integration limits recollect that they must not depend on any inner integration variables. Thus start from outside, i.e. with limits for x. They must not be depend on x (the integration variable) or y and z (the inner variables). Therefore they can only be constant and should be chosen to correspond to the minimum and maximum possible values of the coordinate x, namely, $0 \le x \le 1$. For any fixed value of x the integration in $dzdy$ is performed within a rectangle spanning between $y = 0$ and $y = 1 - x^2$ and $z = 0$ and $z = x$. Thus the required integral is

$$\begin{aligned}
\int_T 4z\,dV &= \int_0^1 \int_0^{1-x^2} \int_0^x 4z\,dz\,dy\,dx \\
&= \int_0^1 \int_0^{1-x^2} 2x^2\,dy\,dx \\
&= \int_0^1 2x^2(1-x^2)\,dx \\
&= \frac{2}{3} - \frac{2}{5} \\
&= \frac{4}{15}
\end{aligned}$$

Other orders of the integration are possible, the intermediate working will differ, but the final result will be the same.

Example 2.12: Apply the divergence theorem to evaluate the surface integral $\int_S \mathbf{F} \cdot d\mathbf{A}$ where $\mathbf{F} = x^2\mathbf{i} - (2x-1)y\mathbf{j} + 4z\mathbf{k}$ and S is the surface of the cone $x^2 + y^2 \le z^2$ and $0 \le z \le 2$.

Solution: Draw the surface as shown below.




```
[x,y]=meshgrid(-2:.2:2);
zc=sqrt(x.^2+y.^2);
```

```
z2=2*ones(size(x));
junk=find(zc>z2);
zc(junk)=repmat(nan,size(junk));
z2(junk)=zc(junk);
mesh(x,y,zc)
hold on
mesh(x,y,z2)
hold off
view(40,35)
```

Thus using T to denote the volume enclosed by the cone

$$\begin{aligned}
\int_S \mathbf{F}\cdot d\mathbf{A} &= \int_T \boldsymbol{\nabla}\cdot\mathbf{F}\,dV \quad \text{by divergence theorem} \\
&= \int_T 2x-(2x-1)+4\,dV \quad \text{computing the divergence} \\
&= \int_T 5\,dV \quad \text{simplifying} \\
&= 5\times(\text{the volume of } T) \\
&= 5\frac{1}{3}2\pi 2^2 \quad \text{using } \tfrac{1}{3}(\text{height})(\text{base area}) \\
&= \frac{40}{3}\pi\,.
\end{aligned}$$

Activity 2.L → Do representative problems from Problem Set 10.7 [Kre06, pp. 463]. Send in to the lecturer for feedback solutions for Q2, 4, 17 and 23 from [Kre06, p. 463].

2.4.1 Volume integral exercises

Gauss' divergence theorem also transforms volume integrals into surface integrals. Such surface integrals may be simpler to evaluate in some way and so make progress. The most obvious simplification in using the divergence theorem to transform a volume integral into a surface integral is that the dimensionality of the integral is reduced. This is a useful feature in some numerical situations. Such a transformation is the basis of using surface integrals to solve Laplace's equation for irrotational and incompressible flow. For example, the flow in the "infinite" domain around a sphere may be reduced to simply solving for special quantities on the sphere's surface.

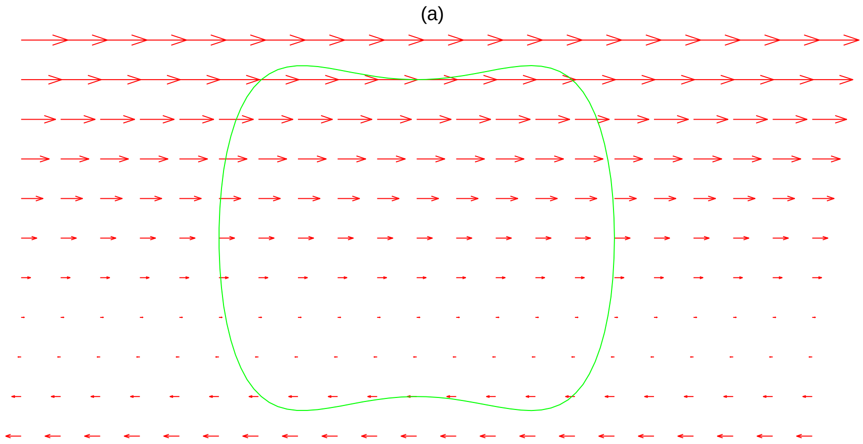
Activity 2.M → Do problems 3–9 from Problem Set 10.8 [Kre06, pp. 468], and the exercises below. Send in to the lecturer for feedback solutions for Q7, 8 from [Kre06, p. 468] and Ex. 2.14 a, b.

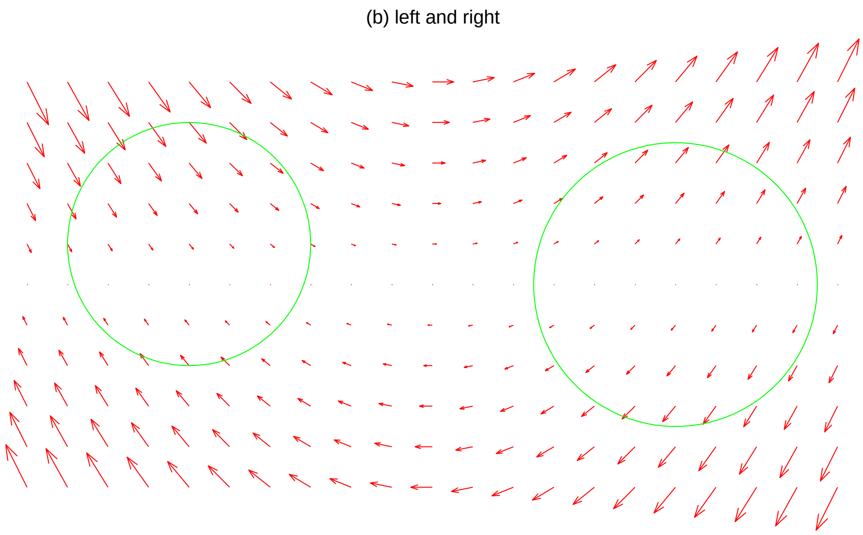
Ex. 2.13: For an incompressible and irrotational fluid flow with velocity potential ϕ, consider $\int_T \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\phi)\,dV$ and argue that the net kinetic energy of the fluid flow in any volume T may be determined from the *surface* integral

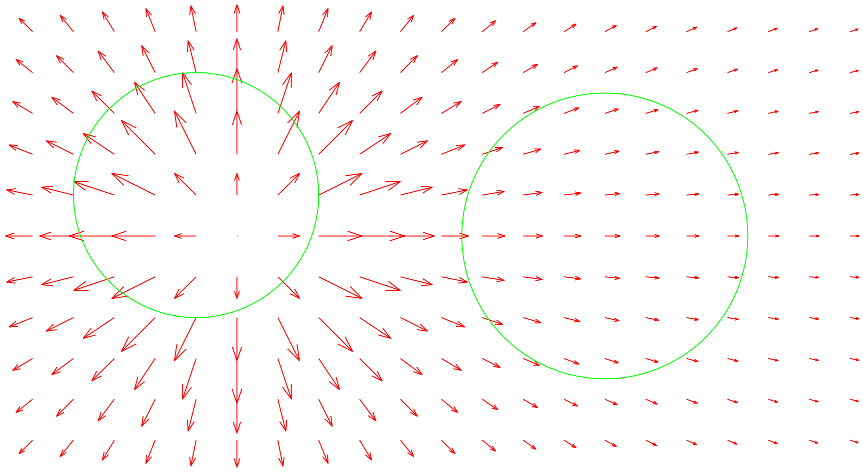
$$KE = \frac{\rho}{2}\int_{S(T)} \phi\frac{\partial\phi}{\partial n}\,dA\,,$$

where $\frac{\partial\phi}{\partial n} = \mathbf{n}\cdot\boldsymbol{\nabla}\phi$.

Ex. 2.14: Consider the fluid velocity fields below and the shown closed curves are a two-dimensional cut through three-dimensional velocity fields and closed cylindrical surfaces. Estimate whether the net fluid flow *out* of the enclosed region, $\int_S \mathbf{v}\cdot d\mathbf{A}$, is positive, negative or approximately zero. Using Gauss' divergence theorem, do these estimates agree with your identification of regions of positive, negative and zero divergence that you made in §§1.3.2?







2.4.2 Conservation principles develop mathematical models

Reading 2.N → Study Examples 1 and 2 in §10.8 of Kreyszig [Kre06, pp. 463–465]. Also peruse Theorem 2 on page 462 as it will be used in the next module.

Activity 2.O → Send in to the lecturer for feedback solutions for Exercise 2.15.

Exercise 2.15: Derive the three-dimensional continuity equation for fluid flow, possibly compressible, that

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \mathbf{v}) = 0\,,$$

using the principle that mass is conserved, and Gauss' divergence theorem over an arbitrary volume T. The argument is very similar to Example 2 [Kre06, pp. 464–465], and to the following derivation of Euler's equation.

Now we similarly go through a derivation of Euler's equation (1.17) using conservation of momentum and Gauss' divergence theorem. For any fixed volume T of the fluid consider the processes that change momentum in T either by being carried into T across its surface S or by body forces applied to all of the fluid in T. First consider the momentum in the vertical direction.

- ρ is the material density and w is the velocity in the vertical z-direction, and so the vertical momentum density is ρw.
- Thus the total vertical momentum inside T is $p_z = \int_T \rho w\, dV$. Its rate of change

$$\frac{dp_z}{dt} = \int_T \frac{\partial}{\partial t}(\rho w)\, dV$$

as T is fixed and so the time derivative may be taken inside the volume integral.

- An applied body force such as gravity, of strength g in the negative z-direction say, acts upon each and every part of the fluid in T in proportion to its density ρ. Thus the net rate of change of momentum due to gravity is the volume integral

$$\int_T -\rho g \, dV \, .$$

- Pressures also act across the surface of T to change the momentum. Across an element dA of the surface S of T, with *outward* unit normal $\mathbf{n}$, the pressure force is $-p\mathbf{n}\,dA = -p\,d\mathbf{A}$ (the negative sign indicates pressure is exerted inwards). But we are only interested, for the moment, in the component in the z-direction, namely $\mathbf{k}\cdot(-p\,d\mathbf{A})$. Thus the contribution to a change in momentum due to pressure forces across the surface totals

$$\begin{aligned} \int_{S(T)} -(p\mathbf{k})\cdot d\mathbf{A} &= \int_T -\nabla\cdot(p\mathbf{k})\,dV \quad \text{by divergence theorem} \\ &= -\int_T \frac{\partial p}{\partial z}\,dV \quad \text{by definition of divergence.} \end{aligned}$$

- The fluid moves with a velocity $\mathbf{v}$ and so the z-momentum density, ρw, is carried around with flux $\mathbf{v}\rho w$. The rate at which z-momentum is thus carried *into* T is thus the surface integral (negative because the normal $\mathbf{n}$ is taken to point *out* of T)

$$\int_S (-\mathbf{v}\rho w)\cdot d\mathbf{A} = -\int_T \nabla\cdot(\mathbf{v}\rho w)\,dV \quad \text{by divergence theorem.}$$

- The principle of conservation of momentum asserts that momentum can only change by these identified processes and so the first expression must equal the sum of the three processes:

$$\int_T \frac{\partial}{\partial t}(\rho w)\,dV = \int_T -\rho g\,dV - \int_T \frac{\partial p}{\partial z}\,dV - \int_T \nabla\cdot(\mathbf{v}\rho w)\,dV\,.$$

  Putting all terms on the left-hand side and writing them all within the one integral shows that

$$\int_T \frac{\partial}{\partial t}(\rho w) + \nabla\cdot(\mathbf{v}\rho w) + \frac{\partial p}{\partial z} + \rho g\,dV = 0\,.$$

  But T is *any* arbitrary volume, so the integrand must be zero everywhere. Thus

$$\frac{\partial}{\partial t}(\rho w) + \nabla\cdot(\mathbf{v}\rho w) + \frac{\partial p}{\partial z} + \rho g = 0\,.$$

- This is a differential equation expressing conservation of momentum in the z-direction. But putting the last two terms on the right-hand side, and expanding some of the left-hand side leads to

$$\rho\frac{\partial w}{\partial t} + w\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})w + \rho\mathbf{v}\cdot\nabla w = -\frac{\partial p}{\partial z} - \rho g\,.$$

From the continuity equation (1.5) the middle two terms on the left-hand side vanish, then dividing by the density results in

$$\frac{\partial w}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} w = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\,.$$

This last equation is the z-component of Euler's equation. Similar arguments derive the other components of Euler's equation. Thus putting them all together we write as before:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p - g\mathbf{k}\,.$$

Exercise 2.16: Similarly derive the x-component of Euler's equation from the principle of conservation of x-momentum and using Gauss' divergence theorem.

2.5 Vorticity and circulation are related by Stokes' theorem

In §§1.5.2 we asserted that irrotational velocity fields must have a velocity potential. Also earlier, §§2.1.2, we asserted that circulation around a curve is exactly the same as the integral of the vorticity over a corresponding surface. It is now time to deliver the support for these assertions in the shape of Stokes'[3] theorem.

The main aim of this section is to show how and why Stokes' theorem transforms some surface integrals into line integrals, and vice-versa.

Stokes' theorem remarkably relates some integrals over a surface S to a line integral around its edge, denoted by the closed curve C:

$$\int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{A} = \oint_C \mathbf{F} \cdot d\mathbf{r}\,. \tag{2.10}$$

In an application to fluid mechanics the vector field may be the velocity field $\mathbf{v}$ and then the curl becomes the vorticity $\boldsymbol{\omega}$. Thus Stokes' theorem asserts that for fluid flow

$$\int_S \boldsymbol{\omega} \cdot d\mathbf{A} = \oint_C \mathbf{v} \cdot d\mathbf{r}\,.$$

That is, the integral of the normal vorticity over some surface is just the circulation around the edge of that surface.

Reading 2.P → Study §10.9 [Kre06, pp. 468–473].

Now prove that any irrotational vector field $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ must have a scalar potential ϕ. The plan is to define a function ϕ as an integral of $\mathbf{v}$, show it is unique by Stokes' theorem, then show it is a potential for $\mathbf{v}$.

[3] George Gabriel Stokes (1819–1903) established the mathematical foundation of many aspects of hydrodynamics. For examples, he derived the velocity of a small sphere moving through a viscous fluid, and obtained solutions for the propagation of nonlinear waves on the surface of water.

- First pick upon any convenient point $P(x_0, y_0, z_0)$ as a reference point. Then for any point $Q(x, y, z)$ in the domain construct a path C from P to Q and define the function

$$\phi(x, y, z) = \int_C \mathbf{v} \cdot d\mathbf{r}\,.$$

- To show that ϕ is independent of the path C joining P to Q consider any two such paths C_1 and C_2. Let a third path C_3 be traversed C_1 forwards and followed by C_2 in the reverse direction so that C_3 is a closed curve starting and finishing at P. Then letting S be any surface with C_3 as its edge,

$$\begin{aligned}
\int_{C_1} \mathbf{v} \cdot d\mathbf{r} - \int_{C_2} \mathbf{v} \cdot d\mathbf{r} &= \oint_{C_3} \mathbf{v} \cdot d\mathbf{r} \\
&= \int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot d\mathbf{A} \quad \text{by Stokes' theorem} \\
&= 0 \quad \text{as } \boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0} \text{ for an irrotational field}
\end{aligned}$$

  Thus $\int_{C_1} \mathbf{v} \cdot d\mathbf{r} = \int_{C_2} \mathbf{v} \cdot d\mathbf{r}$ for all curves with the same end points and hence $\phi(x, y, z)$ is independent of the path—ϕ only depends upon the end point Q and is thus well defined.

- Now show that ϕ is a scalar potential for $\mathbf{v}$. Consider $\frac{\partial \phi}{\partial x}$ from its definition:

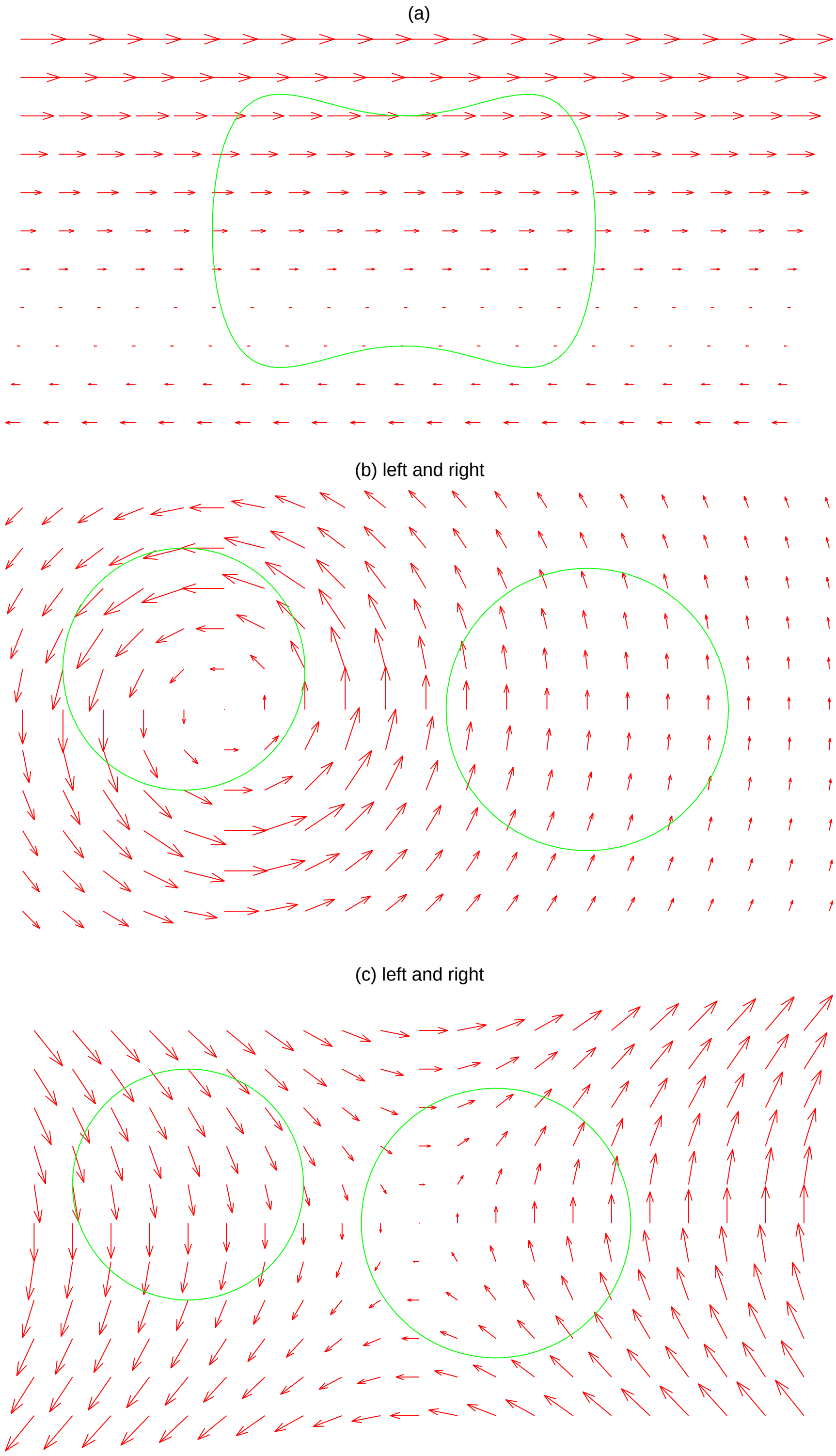
$$\begin{aligned}
\frac{\partial \phi}{\partial x} &\approx \frac{1}{h} [\phi(x + h, y, z) - \phi(x, y, z)] \\
&= \frac{1}{h} \left[\int_P^{(x+h,y,z)} \mathbf{v} \cdot d\mathbf{r} - \int_P^{(x,y,z)} \mathbf{v} \cdot d\mathbf{r} \right] \\
&= \frac{1}{h} \int_{(x,y,z)}^{(x+h,y,z)} \mathbf{v} \cdot d\mathbf{r} \\
&\approx \frac{1}{h} \mathbf{v} \cdot (h\mathbf{i}) \\
&= u\,.
\end{aligned}$$

  Similarly for the other component of $\boldsymbol{\nabla}\phi$ and hence $\mathbf{v} = \boldsymbol{\nabla}\phi$.

Activity 2.Q → Do problems from Problem Set 10.9 [Kre06, p. 473]. Send in to the lecturer for feedback solutions for Q3 and 14 from [Kre06, p. 473], and your answers to the first exercise below.

2.5.1 Stokes exercises

Ex. 2.17: Consider the fluid velocity fields below and the shown closed curves C (directed anticlockwise). Estimate whether the net fluid circulation $\oint_C \mathbf{v} \cdot d\mathbf{r}$ is positive, negative or approximately zero. Using Stokes' theorem, do these estimates agree with your identification of regions of positive, negative and zero vorticity that you made in §§1.5.3?



Ex. 2.18: You should not only be able to use the Gauss' and Stokes' theorems to transform integrals, you should also be able to choose what methods are most appropriate to use in different circumstances. For *each* of the following integrals *outline* two different ways of analytically evaluating them: you may need to consider transformations based upon any of Stokes' theorem, Gauss' divergence theorem, or

scalar potentials. Evaluate *each* of the integrals via the "quickest" method of your choice.

(a) Consider the line "work" integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ and the curve C goes from $(1,0,1)$ to $(3,1,2)$ along the intersection of $x - 2y = 1$ and $z = 1 + y^2$.

(b) Consider the surface flux integral $\int_S \mathbf{G}\cdot d\mathbf{S}$ where $\mathbf{G} = 2y\mathbf{i} - z\mathbf{j} + (2y-1)\mathbf{k}$ and S is the parabolic bowl $z = x^2 + y^2$ for $0 \le z \le 4$ (take the normal to S to point away from the z-axis).

(c) The line "work" integral $\int_C (-z\mathbf{i} + xy\mathbf{j} + x^2\mathbf{k})\cdot d\mathbf{r}$ where the curve C goes from $(0,1,2)$ to $(1,0,2)$ along the intersection of the plane $x + y = 1$ and $z = xy + 2$.

(d) The surface flux integral $\int_S [(y\cos^2 x + y^3)\mathbf{j} + z(\sin^2 x - 3y^2)\mathbf{k}]\cdot d\mathbf{A}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ (the normal to S points outwards).

(e) Consider $\int_S [2z\mathbf{i} + (x - y - z)\mathbf{k}]\cdot d\mathbf{A}$ where S is the plane triangle with vertices $(1,0,0)$, $(0,2,0)$ and $(0,0,1)$ (with the normal to S pointing upwards).

(f) Consider $\int_S \mathbf{B}\cdot d\mathbf{A}$ where $\mathbf{B} = (x^2 + 2x)\mathbf{i} - z\mathbf{j} + (y+1)\mathbf{k}$ and S is the spherical surface $x^2 + y^2 + z^2 = 4$ (with the normal to S pointing outwards).

2.6 Summary

- Line integrals of a scalar function f over a curve C are evaluated by
$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\left|\frac{d\mathbf{r}}{dt}\right| dt\,,$$
when C is parameterised by $\mathbf{r}(t)$ for $a \le t \le b$ (§§2.1.3).
- Line integrals of a vector function $\mathbf{v}$ over a curve C, often called work integrals, are evaluated by
$$\int_C \mathbf{v}\cdot d\mathbf{r} = \int_a^b \mathbf{v}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt\,,$$
when C is parameterised by $\mathbf{r}(t)$ for $a \le t \le b$ (§2.1).
- The circulation around a closed curve C in a fluid flow is the line integral of the velocity field: $\Gamma = \oint_C \mathbf{v}\cdot d\mathbf{r}$ (§§2.1.2).
- If a scalar potential exists for any vector field, then line integrals of the vector field are independent of path—independent provided the path may be continuously deformed while staying entirely within the strict domain of the vector field (§2.2).
- General surfaces may be described by a vector function of two parameters, for example $\mathbf{r}(u,v)$. Then at each point of the surface,

tangent vectors to the surface are $\mathbf{r}_u$ and $\mathbf{r}_v$, and thus a normal to the surface is $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ (§§2.3.2).

Quite often in Cartesian coordinates a surface may be parameterised by two of the coordinates: for example, if $z = f(x, y)$ then $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$ and all the general formula apply to this specific class of parameterisations.

- Over a surface S parameterised by $\mathbf{r}(u, v)$ for $(u, v) \in R$ the surface integral of a scalar function G is computed by

$$\int_S G\,dA = \iint_R G\,|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv\,.$$

These are used to compute the surface areas, mass, centres and moments of thin curved laminae (§§2.3.3).

- Over a surface S parameterised by $\mathbf{r}(u, v)$ for $(u, v) \in R$ the surface integral of the normal component of a vector function $\mathbf{F}$ is computed as

$$\int_S \mathbf{F} \cdot d\mathbf{A} = \iint_R \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv\,.$$

These are used to compute the flux of something through S (§§2.3.3).

- Integrals over a volume in space, similar to double integrals in the plane, may be directly evaluated by three nested single variable integrals (§2.4). For example, if the volume is described by $g(x, y) \leq z \leq h(x, y)$ for $p(x) \leq y \leq q(x)$ for $a \leq x \leq b$, then

$$\int_T f\,dV = \int_a^b \left\{ \int_{p(x)}^{q(x)} \left[\int_{g(x,y)}^{h(x,y)} f(x, y, z)\,dz \right] dy \right\} dx\,.$$

- Gauss' divergence theorem relates some surface integrals to integrals over the volume enclosed by the surface:

$$\int_{S(T)} \mathbf{F} \cdot d\mathbf{A} = \int_T \boldsymbol{\nabla}\cdot\mathbf{F}\,dV\,.$$

This result is used to transform volume integrals for mass, centre of gravity, kinetic energy, etc into surface integrals (§2.4). Conversely, it is used to derive mathematical models by transforming surface processes into volume integrals which then determine specific terms in the model (§§2.4.2).

- Stokes' theorem relates some surface integrals to line integrals around the edge of the surface (§2.5):

$$\int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{A} = \oint_C \mathbf{F} \cdot d\mathbf{r}\,.$$

This result connects vorticity and circulation, and completes the discussion of scalar potentials, line independence and the curl.

II. Mathematical Modelling of Viscous Fluid Flows

Module 3 Viscosity and the Navier-Stokes Equations

Module contents

This module introduces the Navier-Stokes equations as a major model of viscous fluid flows. The rigorous derivation of these equations is very complicated and requires the knowledge of mathematical tools which you are not familiar with (e.g. tensor analysis). This will not be required from you. Instead you should simply understand the physical reasoning behind such a derivation. As you study this module you should focus on the following

Module objectives. You should learn how to

1. write the appropriate form of the Navier-Stokes and continuity equations;
2. impose the correct number of appropriate boundary conditions;
3. non-dimensionalise the equations using the appropriate flow and geometry characteristics;
4. recognise various symmetries (rotational, translational) which are present in a particular flow situation and the notion of steady/unsteady flows to use them for simplifying the Navier-Stokes equations so that you can obtain some exact solutions;
5. understand the physical meaning of the nondimensional parameters and various terms in the equations.

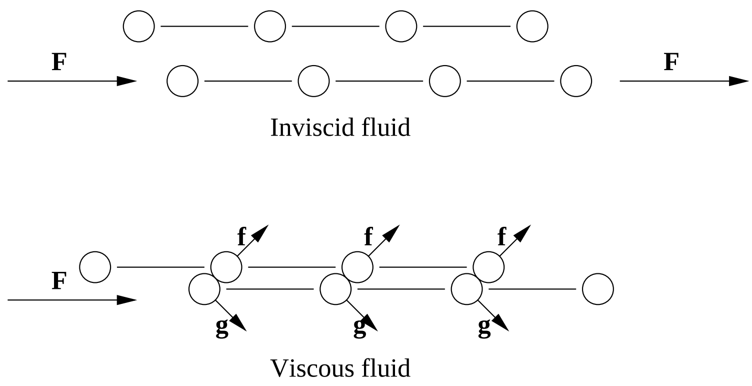


Figure 3.1: Interaction of fluid layers in inviscid (top) and viscous (bottom) fluids.

3.1 Viscosity

Models derived in Part I describe flows of inviscid fluids. In other words, *friction forces* acting in fluids were neglected so far. This approximation works well in very large (ideally unbounded) flow domains, but it does not explain the well known experimental fact that fluid "gets stuck" in the vicinity of the solid walls. For example, we have to wash a car because the air flow cannot sweep away fine dust from its surface no matter how fast we drive.

Objectives:

- to understand the physical origin of viscosity;
- to realise that the force applied to a viscous fluid in one direction may be transmitted in other directions by molecular interactions; and
- to distinguish between Newtonian and non-Newtonian fluids.

Reading 3.A → [OO95], pp. 1–3

The fluid *viscosity* is a result of intermolecular forces, collisions and mixing. At a very simplistic level, we can imagine inviscid fluid as a set of molecular layers which can slide with respect to each other without any interaction as seen from the top cartoon in Figure 3.1. The force applied to such fluid retains its direction throughout the fluid. Now imagine that molecular layers are allowed to collide as shown in the bottom cartoon (you can use the pool balls collision analogy). This would lead to two effects. First, the force of finite amplitude should be applied to a fluid layer to displace it with respect to the neighbouring layers. Second, the force applied in one direction (say from the left) will induce molecular collisions which, subsequently, will result in the forces orthogonal to the initial force (upward and downward forces as seen from Figure 3.1). Experiments (see Figure 1.1 in [OO95, p. 3]) show that for a wide variety of fluids, the force needed to displace one fluid layer with respect to another is proportional to the relative displacement velocity and inversely proportional to the distance between the layers i.e. $\sim \mu \frac{\partial u}{\partial y}$. The coefficient of proportionality μ is called the *dynamic viscosity* of the fluid. It is frequently more convenient to use another coefficient of proportionality $\nu = \mu/\rho$ for incompressible fluids. This is called the *kinematic viscosity*. The dynamic viscosity is a physical property of a given fluid and is measured experimentally. It depends generally on the temperature and the pressure. If μ does not depend on velocity or its gradient, such fluid is referred to as *Newtonian fluid*. Air, water, oil are Newtonian fluids; toffee, chewing gum, melted glass are not. The *non-Newtonian fluids* reveal nonlinear dependence between the velocity gradient and applied force. We will not consider non-Newtonian fluids in this unit.

Activity 3.B → Watch the video "Fluid Dynamics of Drag, Part II" [FDO] ([CD08], Viscosity). Understand what viscosity is and how it is measured (viscosimetry).

3.2 Navier-Stokes equations

The Euler equations derived in Section 1.6 cannot be used to model flow of a realistic viscous fluids. They must be extended to account for a new physical mechanism associated with molecular motion—*viscous dissipation*. The most complete model for fluid flows is the system of the *Navier-Stokes equations*.

Objective:

- to understand the physical reasoning behind the major steps in the derivation of the Navier-Stokes equations.

Reading 3.C → [OO95], pp. 3–11; [MAT], Green's Theorem

The obvious disadvantage of the Euler equations is that they cannot account for the viscous forces. Thus all one needs to derive a complete system of the Navier-Stokes equations is to add terms modelling viscous effects to the Euler equations. This is the subject of section 1.2 in [OO95]. You might find the following comments useful when studying it.

- The *summation convention*: if the same index is used in the product twice it means that the summation with respect to this index is assumed. For example, if $i = 1, 2$ then $x_{ij}x_{ik} = x_{1j}x_{1k} + x_{2j}x_{2k}$.
- Make sure that you understand the meaning of subscripts in σ_{ij}: i is the direction of force, j is the direction normal to the surface to which the force is applied.
- Consider a cube of size a. Take $a \to 0$. Show that the volume of the cube tends to 0 faster than its surface area. Relate this fact to the derivation of equation (1.3) in [OO95, p. 4], namely, justify why the right-hand side of the first equation on p. 4 in [OO95] is assumed to be zero while the left-hand side is still non-trivial.

- You may omit discussion of tensor operations and transformations.
- The *Newton's law of conservation of angular momentum* states that the *angular acceleration* about the given point (i.e. the acceleration of rotation about this point) is proportional to the sum of all angular momenta about this point. *Angular momentum* is the product of the force and the distance from the point to the line along which the force acts.
- Equation (1.9) in [OO95, p. 7] can be foreseen based on assumptions that the fluid is Newtonian and that the *stress tensor* is symmetric. The first assumption gives an idea of the functional form of d_{ij}: all sorts of velocity gradients lead to shear forces and *shear stresses* which are proportional to these gradients. Consequently, d_{ij} must be a linear combination of $\frac{\partial u_i}{\partial x_j}$, $i, j = 1, 2, 3$. The second assumption defines the indexes. Since d_{ij} must be the same as d_{ji}, the only possible combinations are $d_{ij} \sim \lambda \frac{\partial u_i}{\partial x_j}\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, where λ and μ are properties of fluid and thus cannot depend on i, j.
- The *bulk viscosity* λ appears only in compressible fluids. Imagine that molecules which belong to the same fluid layer (see Figure 3.1) collide with each other rather than with those belonging to the other layers. This would lead to a resistance force as we try to change the distance between the molecules when, say, compressing the fluid. In incompressible fluids the average distance between the molecules cannot be changed and thus the bulk viscosity is 0. In compressible fluids the *Stokes' assumption* that $\lambda = -(2/3)\mu$ is usually used.
- The Navier-Stokes equations for compressible fluid with constant viscosity in a Cartesian coordinate system (x, y, z) are

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{3.1}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \tag{3.2}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right), \tag{3.3}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0, \tag{3.4}$$

where the modified pressure P which includes effects of compressibility of a viscous fluid is

$$P = p + (\lambda + \mu)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right).$$

Activity 3.D → Watch the video “Fluid Dynamics of Drag, Part II” [FDO] ([CD08], Viscosity). Distinguish between normal and tangential forces. Send in to the lecturer for feedback solutions for Exercises 3.1–3.4.

3.2.1 Exercises

Ex. 3.1: Give the definition of stress. What do the terms “normal stress” and “shear stress” mean?

Ex. 3.2: Explain where exactly the concept of Newtonian fluid is used in deriving the Navier-Stokes equations.

Ex. 3.3: Formulate the Green’s theorem which *is used* in derivation of equation (1.19) in [OO95].

Ex. 3.4: Use the summation convention to show how equation (1.20) is obtained from equation (1.19) (see [OO95, p. 10]). Then assume that $\mu =$ const. and write the equation for u_1 using (u_1, u_2, u_3) and (x_1, x_2, x_3) explicitly. Finally obtain equation (1.21) in [OO95, p. 11].

3.3 Vorticity revisited

As introduced in Section 1.5 the quantity $\omega \equiv \nabla \times \mathbf{v}$ is called *vorticity*.

Objectives:

- to review the physical meaning of vorticity;
- to derive an evolution equation for vorticity in viscous fluid;
- to understand the concept of vorticity diffusion in 2D viscous flows;
- to understand the concept of vorticity stretching in 3D flows.

Reading 3.E → [OO95], pp. 13–14; [Kre06], Appendix A3.4 and pp. A71–A73

Equation (1.25) in [OO95, p. 13] is obtained by applying the *curl* operator to the vector form of the Navier-Stokes equations. For two-dimensional flows it becomes

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega \,, \tag{3.5}$$

where D denotes the material derivative. Consider pure circular motion[1] of the fluid whose velocity given initially by

$$v_{\theta 0} = \frac{\Gamma}{r} \,, \quad r > 0 \,, \tag{3.6}$$

[1] The line parallel to the vorticity vector at every its point is called the *vortex line*. The vortex line may be carried away by the flow, but we always can choose a moving coordinate system relatively to which the vortex line does not drift.

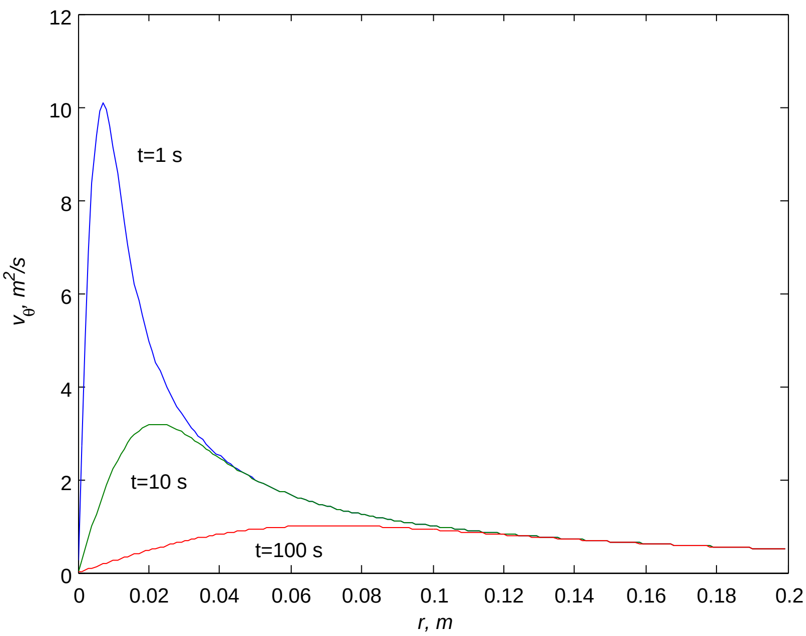


Figure 3.2: Velocity in rotating air ($\nu = 10^{-5}$ m^2/s) for different times for $\Gamma = 0.1$ m^2/s.

where Γ is constant and r is the radial distance from the centre of rotation. It is convenient to rewrite equation (3.5) in *polar coordinates* using the fact that $v_r = v_z = \partial/\partial z = \partial/\partial\theta = 0$ (see [Kre06, A3.4, pp. A71–A73])[2]

$$\frac{\partial\omega}{\partial t} = \nu\left[\frac{\partial^2\omega}{\partial r^2} + \frac{1}{r}\frac{\partial\omega}{\partial r}\right] , \tag{3.7}$$

which is the *diffusion equation* in polar coordinates. Its solution is

$$\omega(r,t) = |curl\,\mathbf{v}| = \frac{1}{r}\frac{\partial(rv_\theta)}{\partial r} = \frac{\Gamma}{2\nu t}e^{-\frac{r^2}{4\nu t}} , \tag{3.8}$$

where

$$v_\theta(r,t) = \frac{\Gamma}{r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right) . \tag{3.9}$$

For any time $t > 0$, the vorticity is 0 at the origin since it is terminated by the viscous dissipation instantly. For any fixed distance $r > 0$ from the origin, the vorticity is initially 0 (potential flow at $t = 0$) and increases until it reaches its maximum value

$$\omega_{\max} = \frac{2\Gamma}{er^2} \quad \text{at} \quad t = \frac{r^2}{4\nu} . \tag{3.10}$$

At $t \to \infty$ vorticity $\omega \to 0$ everywhere and thus the motion decays. The velocity distribution at different times is shown in Figure 3.2.

[2] Condition $\partial/\partial z = 0$ is sometimes called *translational symmetry* in z, while $\partial/\partial\theta = 0$ is referred to as *rotational symmetry*.

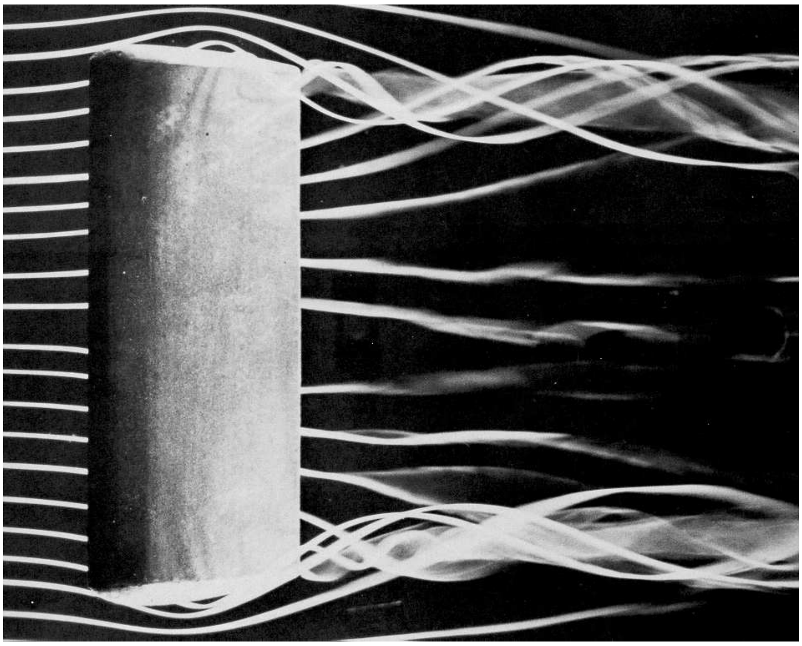


Figure 3.3: (photograph from [Dyk82]). Trailing vortices from a rectangular wing. The centres of the vortex cores spring from the trailing edge at the tips. The model is tested at Reynolds number 100000.

Thus in 2D flow of viscous fluid the vorticity spreads slowly (diffuses) through the fluid and eventually decays.

Activity 3.F → Watch the video "Fluid Flows" [FF85] and [CD08], Vorticity. Make a note that vorticity is concentrated near the centre of rotation thus the flow is nearly potential away from the centre. Send in to the lecturer for feedback solutions for Exercises 3.5–3.8.

To better interpret what can happen to the vorticity in three-dimensional flows, imagine two "marked" fluid particles initially located at the points on the same vortex line (the magnitude of vorticity is the same at both points). Refer to Figure 3.3 to visualise better the concept of the vortex line: the flow is irrotational in front of the air-foil which creates vorticity behind it. The vortex lines are the centre lines of the vortices originating at the air-foil tips. If the flow is so that the distance between the "marked" fluid particles increases (decreases) as they travel with the flow, the magnitude of vorticity at the corresponding points will increase (decrease) proportionally. In other words, the rotation speed about the line connecting these fluid particles will increase (decrease). Imagine a Figure skater lifting his hands above his head as he spins to see the similarity. This is called the *vorticity stretching*. Thus in three-dimensional flows the two effects could compete: *vorticity decay* due to diffusion and vorticity amplification due to stretching. But stretching cannot be infinite since all realistic flow domains have finite sizes while the diffusion is independent of the size of the flow region. Thus eventually all 3D vortical flows will decay unless the energy for its support is constantly supplied from an external

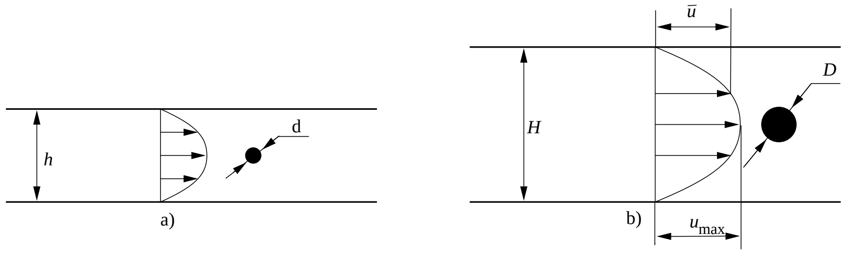


Figure 3.4: Flow similarity.

source (fan, aircraft propeller, etc.).

3.3.1 Exercises

Ex. 3.5: Use expressions for operators *grad* and *curl* in polar (cylindrical) coordinates given in [Kre06, A3.4, A71–A73] and conditions of rotational symmetry to derive equation (3.7) starting with equation (3.5).

Ex. 3.6: Use rotational symmetry in polar coordinates to show that $|curl\,\mathbf{v}| = \frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}$. Then show by direct substitution that (3.8) satisfies equation (3.7).

Ex. 3.7: Use some graphics software (e.g. MATLAB) to plot vorticity (3.8) as a function of r for the same times as in Figure 3.2.

Ex. 3.8: Show how result (3.10) is obtained.

3.4 Nondimensionalisation

Objectives: to understand

- the importance of non-dimensionalisation in modelling fluid flows;
- the role of non-dimensional parameters;
- how to choose proper characteristic flow quantities for nondimensionalisation.

Reading 3.G → [OO95], pp. 14–15

There exists enormous variety of fluid flows in nature. Many of them can be described by the Navier-Stokes equations. The question then arises whether the solution obtained for one flow (Figure 3.4a)) can be used somehow to provide a solution for a similar but different in details flow (Figure 3.4b)) without recomputing the flow from the very beginning. And the answer is yes. If the equations are non-dimensionalised properly then their solution is somewhat generic in the sense that a whole family of physical (dimensional) solutions can be obtained by simply multiplying the *non-dimensional solutions* by characteristic (dimensional) quantities (e.g. mean flow velocity, width of the channel etc.) provided that a certain combination of them called *non-dimensional parameters* (e.g. the *Reynolds* *number*) is the same for both flows. This saves a lot of computational efforts! The flows with the equal Reynolds numbers $Re = \frac{\rho V_0 L}{\mu}$ are called *dynamically similar*.

Non-dimensional parameters emphasise the importance of different physical forces in a given flow. For example, the Reynolds number is the ratio of inertia and viscous forces: if Re is large (large size or fast moving bodies) it means that the flow is essentially inviscid since viscous forces are negligible in comparison with inertia and vice versa. Thus the non-dimensional parameters give a clue as to which terms in the Navier-Stokes equations can be neglected to obtain an approximate but sufficiently accurate solution for a given flow.

An additional advantage which nondimensionalisation provides is the possibility of "zooming" into the fine flow structures. For example, when considering the channel flow away from the ball in Figure 3.4 one of the practically important characteristics is the flow rate q. Thus it is convenient to non-dimensionalise the equations using the channel width H and the average flow velocity $\bar{u}$. Then the non-dimensional flow rate $q' = \frac{q}{H\bar{u}}$ will be of the order 1 which is convenient to tabulate for practical use. On the other hand if we are interested in a detailed flow around the ball, and in particular in the drag force acting on the ball, then the proper characteristic quantities are diameter of the ball D and the maximum velocity of the flow $u_{\max}$ since the ball is located near the centre line of the channel. After such a nondimensionalisation, the flow region of interest in the close vicinity of the ball will have size of the order 1 which is again convenient to analyse.

Activity 3.H → Watch the video "Fluid Dynamics of Drag, Part II" [FDO] ([CD08], Reynolds Number). Make a note about the dynamic similarity of the flows around a very small ball dropping in the water and around a huge meteorological balloon rising in the air. Send in to the lecturer for feedback solutions for Exercises 3.9 and 3.10.

3.4.1 Exercises

Ex. 3.9: Let the kinematic viscosity of the fluid in Figure 3.4 be ν. Introduce and compare the Reynolds numbers for Figures 3.4 a) and b) assuming that the average fluid velocities are the same. Recollect the physical meaning of the Reynolds number and, based on that, deduce which channel requires a more powerful pump to maintain a steady laminar flow.

Ex. 3.10: Consider Figure 3.4 b). Introduce and compare the proper Reynolds numbers for the average channel flow and for the flow around the ball. Deduce in which case the viscous forces are more important. Assume that $D \ll H$.

3.5 Some exact solutions

Objectives:

- to practice non-dimensionalising the Navier-Stokes equations;
- to learn how to impose physical boundary conditions and to non-dimensionalise them;
- to understand the concept of fully developed flow.

3.5.1 Plane Couette flow

Reading 3.I → [OO95], pp. 3 and 15, problem 12, pp.18–19; ([Sch79], Chapter V, pp. 91-93)

We consider a two-dimensional flow shown in Figure 1.1 in [OO95, p. 3]. We assume that fluid is incompressible and its viscosity is constant. The upper plate is instantly brought in motion with the horizontal velocity U at $t = 0$. Start with the set of dimensional Navier-Stokes equations, however for further convenience[3] and in contrast to the text book denote all dimensional quantities by primes. We non-dimensionalise the flow quantities as follows

$$u' = Uu\,, \quad v' = Uv\,, \quad x' = hx\,, \quad y' = hy\,, \quad t' = \frac{h}{U}t\,, \quad p' = \rho U^2 p\,. \tag{3.11}$$

Substitute (3.11), $\rho = \text{const.}$, $w' = \frac{\partial}{\partial z'} = 0$ and $\mathbf{g} = (g_x, g_y, g_z) = (0, -g, 0)$ into (3.1)–(3.4) and cancel common factors to obtain the non-dimensional Navier-Stokes equations describing a two-dimensional flow of incompressible fluid

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{3.12}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{1}{Fr}, \tag{3.13}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{3.14}$$

where $Fr = \frac{U^2}{gh}$ is the non-dimensional *Froud number*. The fluid cannot slip along the plates nor can it flow through them. Thus the non-dimensional boundary conditions become

$$u(0,t) = v(0,t) = v(1,t) = 0\,, \quad u(1,t) = 1 \quad \text{for } t > 0\,. \tag{3.15}$$

We assume plates of infinite extent and thus flow profile does not depend on the longitudinal coordinate x (the so-called *translational symmetry*). Thus $\frac{\partial}{\partial x} = 0$ and then from the continuity equation (3.14) using boundary conditions (3.15) we immediately obtain that $v(x, y, t) = 0$. Then

[3] In the subsequent sections we will always work with non-dimensional Navier-Stokes equations unless explicitly noted otherwise. For this reason from now on we will use primes to denote dimensional quantities.

integration of (3.13) gives $p = -\frac{y}{Fr} + \text{const.}$ and equation (3.12) becomes

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}\,. \tag{3.16}$$

First, find the steady solution from $\frac{\partial^2 u}{\partial y^2} = 0$ and boundary conditions (3.15). It is $u_s(y) = y$. The complete solution is then $u = u_s(y) + u_u(y,t) = y + u_u(y,t)$, where

$$u_u(0,t) = u_u(1,t) = 0\,. \tag{3.17}$$

Let $u_u(y,t) = T(t)Y(y)$ (this is called the *method of separation of variables*). Substitute it in (3.16) and divide by u_u to obtain

$$\frac{T'(t)}{T(t)} = \frac{1}{Re}\frac{Y''(y)}{Y(y)} = -K^2 = \text{const.}$$

Both ratios must be equal to the same (unknown) constant since the functions forming them depend on different variables. The specific form of the constant is chosen for convenience as will be seen below. Solving for $T(t)$ and $Y(y)$ we obtain

$$T(t) = e^{-K^2 t}\,, \quad Y(y) = A\sin(Ky\sqrt{Re}) + B\cos(Ky\sqrt{Re})\,.$$

It follows from (3.17) that $B = 0$ and the nontrivial solution exists only if $K = K_n = \frac{\pi n}{\sqrt{Re}}$, $n = \pm 1, \pm 2, \ldots$. Without any loss of generality choose $n > 0$. Then the most general solution for u is

$$u = y + \sum_{n=1}^{\infty} A_n e^{-\frac{\pi^2 n^2}{Re}t}\sin(n\pi y)\,.$$

At $t = 0$ velocity $u = 0$, thus

$$-y = \sum_{n=1}^{\infty} A_n \sin(n\pi y)\,.$$

Multiplying the above equation by $\sin(m\pi y)$, integrating from 0 to 1 and using the *orthogonality* property of sin functions

$$\int_0^1 \sin(m\pi y)\sin(n\pi y)\,dy = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & n = m \end{cases}$$

we obtain $A_m = \frac{2(-1)^m}{m\pi}$ and finally

$$u = y + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\frac{\pi^2 n^2}{Re}t}\sin(n\pi y)\,.$$

Note that as $t \to \infty$ the solution exponentially quickly approaches the linear velocity profile shown in Figure 1.1 in [OO95]. This approach is the quickest for very viscous fluids ($Re \to 0$) and the characteristic time for establishing the steady flow is $t_s = \frac{h}{U}\frac{Re}{\pi^2} = \frac{h^2}{\pi^2 \nu}$ (based on the slowest decaying time-dependent mode $n = 1$). This time becomes infinitely large in the inviscid limit $\nu \to 0$ because in that case the upper plate slips along the fluid surface without interacting with it.

Activity 3.J → Send in to the lecturer for feedback solutions for Exercises 3.11 and 3.12.

3.5.2 Exercises

Ex. 3.11: Starting with the dimensional Navier-Stokes equations, use nondimensionalisation (3.11) to obtain equations (3.12)–(3.14).

Ex. 3.12: Show in detail how coefficients A_m are obtained for the Couette flow.

3.5.3 Recommended procedure for solving the Navier-Stokes equations

The example of the previous section contains a number of general steps which we summarise below providing the references to a Couette flow parenthetically. These general steps should be followed in order to make the solution procedure short, yet systematic, and to reduce the likelihood of errors.

1. Formulate the physical problem and list any simplifying assumptions made and discuss there implications. (Firstly, the sudden change of the plate velocity from 0 to U is unphysical as it implies infinite acceleration $a = du/dt$ and therefore infinite force $F = ma$, where m is the mass of the upper plate. Since both force and mass are always finite so is the acceleration and therefore it will take finite time for the upper plate to start moving with velocity U. However we assume that this time is small enough to be neglected or, alternatively, we are only interested in the long term solution at times much larger than the plate acceleration time. Secondly, we assume infinite extent of both plates in the x and z directions. This is unrealistic. Therefore the solution which we obtain is only relevant in the region between large but finite planes far away from their edges where the end effects are negligible. Thirdly, the density, viscosity and gravity are constant. These conditions restrict us to solutions for flows of incompressible viscous fluid in a uniform gravitational field—very wide class of realistic flows, however not completely arbitrary.)

2. Sketch the problem geometry, indicate major dimensions, identify the boundaries. Introduce an appropriate coordinate system which follows the boundaries. (Cartesian system with x along the lower plate in the direction of motion, y chosen upwards and z chosen to form a right-hand coordinate system.)

3. Identify any spatial symmetries i.e. invariances with respect to translation or rotation in certain directions. These symmetries lead to vanishing derivatives in the corresponding directions. (Translational symmetry along the plates leads to $\partial/\partial x = \partial/\partial z = 0$.)

4. Use the initial and boundary conditions to identify possible form of the solution or of its parts. (Since initially the fluid is at rest and there is no force applied in the z-direction there is no acceleration in this direction. Therefore w will remain zero at all times. Along with the translational symmetry in z this condition justifies reducing the consideration to two spatial dimensions x and y.)

5. Write full dimensional system of the Navier-Stokes *and* continuity equations in a *component* form in the chosen coordinate system. Cross out any terms which are zero because of various symmetries. (Only the x- and y-momentum and continuity equations are necessary for a Couette problem and all $\partial/\partial x$ terms are zero.)

6. Examine the order of derivatives in the remaining terms: it is equal to the number of the boundary conditions you need to specify for each of the unknown functions. Provide algebraic description of each of the boundaries and specify the appropriate physical boundary conditions. (For example, the dimensional form of equation (3.12) involves non-vanishing $\partial^2 u/\partial y^2$. This means that two boundary conditions have to be specified for u for the specified values of y. The algebraic dimensional equations for the two boundaries, i.e. the two plates, are $y = 0$ and $y = h$. Therefore the two physical boundary conditions are those of no-slip along the plates: $u = 0$ at $y = 0$ and $u = U$ at $y = h$.)

7. Write the resulting reduced sytem of the dimensional Navier-Stokes and continuity equations along with the required number of the boundary/initial conditions.

8. Examine the geometry of the problem, its initial and boundary conditions to determine suitable parameters to be used as scales (h, U, ρ, μ).

9. Use these scales to non-dimensionalise the derived equations *and* initial/boundary conditions. Identify all non-dimensional parameters which appear (Re and Fr). Realise that the total number of non-dimensional parameters is always smaller than that of physical ones (Re and Fr vs U, h, ρ, μ, $\mathbf{g}$) and therefore non-dimensional equations are easier to solve.

10. Solve the resulting non-dimensional equations using the appropriate mathematical technique (e.g. separation of variables).

11. Use the chosen scales to convert the non-dimensional quantities back to dimensional to obtain a physical solution to a problem.

The above steps are given in a slightly different order from what is used in the previous or the following sections, but the major steps are easily recognised in both.

3.5.4 Fully developed plane Poiseuille flow

Reading 3.K → ([Whi94], pp. 327–329; [Sch79], Chapter V, pp. 83-85)

Now let us consider flow between two long and wide resting parallel plates. In order to maintain such a flow, a pressure gradient must be applied along the channel. Let us consider the flow region far away from the edges of the plates. If the applied pressure gradient is steady then the flow becomes *fully developed* after all transients associated with, say, switching the pump on decay. The velocity profile in such a flow is steady and does not depend on the location along the channel. The steady solution for the Couette flow considered in Section 3.5.1 is another example of fully developed flow. We use (3.11) to non-dimensionalise equations again except the characteristic velocity U is yet unknown. At this stage we introduce it formally. We will choose the appropriate value for it after we get an idea on what the actual velocity profile is. Since for a fully developed flow $\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{v}}{\partial x} = 0$, as in the previous section, we obtain $v = 0$ and $p = -\frac{y}{Fr} + p_1(x)$. Note that pressure is not constant along x anymore to maintain the flow. Equation (3.12) then becomes

$$\frac{dp_1(x)}{dx} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} \tag{3.18}$$

with the no-slip boundary conditions at the walls $u(0) = u(1) = 0$. Since $u \neq u(x)$

$$\frac{dp_1(x)}{dx} = \Pi = \text{const.} \quad \text{or} \quad p_1 = \Pi x + C\,,$$

where Π is some undetermined yet constant. Then integration of (3.18) in y and the no-slip boundary conditions lead to

$$u = \frac{1}{2}Re\Pi y(y-1)\,. \tag{3.19}$$

Now after we know the form of the velocity profile we can choose the characteristic velocity U used in nondimensionalisation. Let, for example, U be a maximum velocity. It is easy to show that the flow velocity (3.19) has a maximum $u_{\max} = -\frac{1}{8}Re\Pi$ when $\Pi < 0$ at $y = \frac{1}{2}$ i.e. at the centre of the channel. Then it follows from (3.11) that the non-dimensional maximum velocity must be 1 and thus $\Pi = -\frac{8}{Re}$. Finally, the solutions are

$$u = 4y(1-y)\,, \quad p = -\frac{y}{Fr} - \frac{8x}{Re} + C\,.$$

Note that the pressure gradient necessary to maintain a certain flow velocity is inversely proportional to the Reynolds number i.e. proportional to viscosity and inversely proportional to the width of the channel (recollect Exercise **Ex. 3.9**). It is interesting to find that the the average dimensional flow velocity $\bar{u}$ is

$$\frac{\bar{u}}{u_{\max}} = \frac{\int_0^1 u\,dy}{\int_0^1 dy} = \frac{2}{3}$$

i.e. it is equal to two thirds of the maximum velocity. This result was used in Exercise **Ex. 3.10**.

Activity 3.L → Send in to the lecturer for feedback solutions for Exercise 3.13.

3.5.5 Exercises

Ex. 3.13: Apply the step-by-step procedure outlined in Section 3.5.3 to derive the solution for the plane Poiseuille flow and compare it with the derivation given in Section 3.5.4.

3.6 Summary

- All real fluids (except super-fluids such as He^3 at cryogenic temperatures) are viscous. The viscosity (dynamic μ or kinematic $\nu = \mu/\rho$) is the physical property of the fluid and is a result of molecular collisions and mixing. It usually depends strongly on the temperature and slightly on the pressure, but not on the material of the walls of the container, pipe or channel. Because of the viscosity the fluid cannot move along the solid surface. This is called *no-slip condition* for viscous fluid.

- In viscous fluids the force acting on a fluid in one direction can induce forces in other directions due to molecular interactions. The force acting on a unit surface is called *stress*. *Normal* and tangential stresses are distinguished. The fluids in which stresses are proportional to the *rate of shear deformation* (velocity gradients) are called Newtonian. The most complete system of equations describing the flow of Newtonian fluid is the system of the Navier-Stokes equations complemented with the continuity equation
$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \frac{\partial}{\partial x_j}\left(\lambda\frac{\partial u_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right),$$
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0,$$
where indexes i, j, k run from 1 to 3 and correspond to different coordinate axes in a three-dimensional space, D denotes material derivative, λ is the bulk viscosity and the summation convention is used (i.e. if the subscript is repeated then the sum over all possible values of this subscript is assumed). For incompressible fluid with constant viscosity the above equations are simplified to
$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu\nabla^2 u_i,$$
$$\frac{\partial u_i}{\partial x_i} = 0,$$
where $\frac{D}{Dt}$ denotes the material derivative and ∇^2 is the Laplacian operator.

- The proper nondimensionalisation of the equations has to be performed before solving them. This allows one to find a whole family of physically similar solutions using a single computational run as well as to analyse the relative importance of different physical terms in the equations to "zoom" into the fine structures existing in the flow. If the nondimensionalisation is performed correctly all the major components (say, the largest velocity component) of a non-dimensional solution will be of order 1.

- Exact solutions for the Navier-Stokes equations exist only for a limited number of relatively simple laminar flows. Some simplifying assumptions such as incompressible fluid, constant viscosity, simple geometry, steadiness etc. are necessary to obtain these solutions.

Module 4 Boundary Layers in Fast Flows

Module contents

4.1 Asymptotics

Systems of equations such as the Navier-Stokes equations are typically too complicated to be solved analytically. As we have seen in Section 3.5, the exact solutions can only be found for a very limited number of relatively simple flow situations. On the other hand, proper nondimensionalisation enables one to devise a sound approximation to the original equations by retaining only the terms of the highest importance (order). Solutions of such model equations are simpler to obtain and they contain most of the major features of the full solutions. Such solutions are called *asymptotic*.

Objectives:

- to understand when an asymptotic solution can be found;
- to distinguish between the regular and singular perturbation problems;
- to be able to develop inner and outer solutions for a singular perturbation problem; and
- to view an inner solution as a boundary layer.

Reading 4.A → [OO95], pp. 22–28

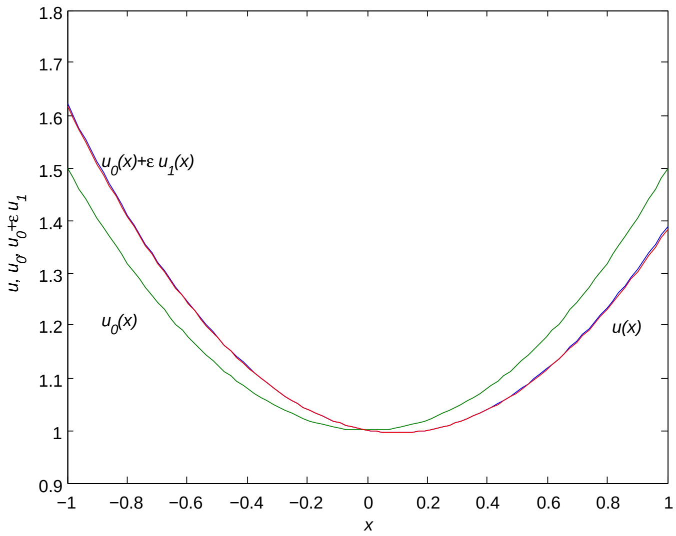


Figure 4.1: Exact and asymptotic solutions of equation (4.1) for $\epsilon = 0.1$. The two term asymptotic solution almost coincides with the exact solution.

4.1.1 Regular perturbation problems

Consider equation

$$\frac{du}{dx} + \epsilon u = x\,, \quad u(0) = 1\,, \quad 0 < \epsilon \ll 1 \tag{4.1}$$

with the exact solution

$$u = \left(1 + \frac{1}{\epsilon^2}\right) e^{-\epsilon x} + \frac{x}{\epsilon} - \frac{1}{\epsilon^2}\,. \tag{4.2}$$

If $\epsilon \to 0$ (we call it the *small parameter*) then we can neglect the ϵu term and solve

$$\frac{du_0}{dx} = x\,, \quad u_0(0) = 1 \tag{4.3}$$

instead of (4.1) to obtain an approximate solution

$$u_0 = \frac{x^2}{2} + 1\,. \tag{4.4}$$

Both solutions are plotted in figure 4.1 for $\epsilon = 0.1$. The approximate solution u_0 is quite simple algebraically in comparison with the exact u, but it is not sufficiently accurate for any finite value of ϵ as seen from figure 4.1. Can we improve u_0 in any systematic way? Yes! Let us look for the approximate solution $\tilde{u}$ in the following form

$$\tilde{u} = u_0 + \epsilon u_1 + \ldots \tag{4.5}$$

Substitute this *asymptotic expansion* into (4.1) to obtain

$$\frac{du_0}{dx} + \epsilon\frac{du_1}{dx} + \epsilon u_0 + \epsilon^2 u_1 + \ldots = x\,, \quad u_0(0) + \epsilon u_1(0) + \ldots = 1\,. \tag{4.6}$$

Now collect terms multiplied by $\epsilon^0, \epsilon^1, \dots$ (equivalently we say, collect terms of order $\epsilon^0, \epsilon^1, \dots$ or collect $\mathcal{O}(1), \mathcal{O}(\epsilon), \dots$ terms) to obtain a *recursive set* of differential equations which are simpler than (4.1). The $\mathcal{O}(1)$ terms result in (4.3) and the $\mathcal{O}(\epsilon)$ terms lead to

$$\frac{du_1}{dx} + u_0 = 0\,, \quad u_1(0) = 0 \quad \text{so that} \quad u_1 = -x\left(1 + \frac{x^2}{6}\right)\,. \tag{4.7}$$

As seen from figure 4.1, a sum $u_0 + \epsilon u_1$ is essentially indistinguishable from the exact solution u and still $u_0 + \epsilon u_1$ has a much simpler (polynomial[1]) form than u!

In the above example, the term other than the highest derivative was multiplied by a small parameter and each of the asymptotic solutions u_0, u_1, ... satisfied imposed boundary conditions. Such problems are called *regular perturbation problems.*

4.1.2 Singular perturbation problems

Now consider equation

$$\epsilon\frac{du}{dx} + u = x\,, \quad u(0) = 1\,, \quad 0 < \epsilon \ll 1 \tag{4.8}$$

with the exact solution

$$u = (1+\epsilon)e^{-\frac{x}{\epsilon}} + x - \epsilon\,. \tag{4.9}$$

Now as in Section 4.1.1, we write

$$u_o = u_{o0} + \epsilon u_{o1} + \dots\,, \tag{4.10}$$

substitute this asymptotic expansion into (4.9) and collect terms at ϵ^0 and ϵ^1 to obtain

$$u_{o0} = x \quad u_{o0}(0) = 1\,, \tag{4.11}$$

$$u_{o1} = -\tfrac{du_{o0}}{dx} = -1 \quad u_{o1}(0) = 0\,. \tag{4.12}$$

Approximate solution

$$u_o = x - \epsilon + \dots \tag{4.13}$$

is called *the outer solution.* As seen from figure 4.2, it is a very good approximation for the exact solution (4.9) in the *outer region* away from the boundary $x = 0$, but none of its components satisfy the imposed boundary conditions. The form of the exact solution (4.9) indicates the presence of a *boundary layer* with very steep gradients where the outer solution fails. This is a direct consequence of neglecting higher order derivatives in the original equation. When doing so, we assumed that $|du/dx| \sim |u| \sim 1$ everywhere, thus, as in Section 4.1.1, we could neglect terms multiplied by ϵ. Apparently this assumption fails near $x = 0$ where $|du/dx| \sim 1/\epsilon$

[1] You may show that this approximate solution is the *Taylor series expansion* in ϵ of the exact solution (Exercise **Ex. 4.1**) truncated after two terms.

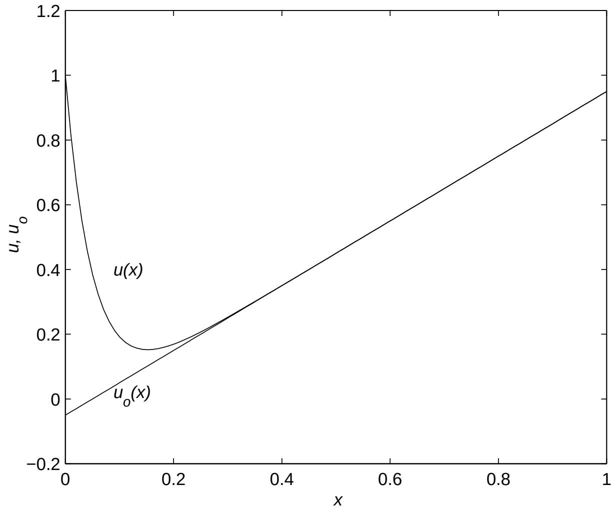


Figure 4.2: Exact and outer solutions ((4.9) and (4.13), respectively) of equation (4.8) for $\epsilon = 0.05$. The inner solution is identical to the exact solution.

(see figure 4.2) and the derivative term cannot be neglected even when it is multiplied by ϵ. In fact

$$\left.\frac{du}{dx}\right|_{x=0} = 2 - \frac{1}{\epsilon} \approx -\frac{1}{\epsilon}$$

and thus $\epsilon du/dx \approx -1$ independently of how small ϵ is. Thus when a small parameter multiplies the term involving the highest derivative the size of ϵ determines the *thickness of the boundary layer*, the layer where the gradients are very steep and the outer solution is not valid no matter how many terms in its asymptotic expansion we retain. Such problems are called *singular perturbation problems*.

Let us consider the boundary layer in more detail. In order to do that, we need to "magnify" it or "zoom into" it by stretching our coordinate x near 0. Let us introduce a new variable $X = x/\epsilon$. It is referred to as the *inner* or *stretched coordinate* while x is called the *outer coordinate*. If, for example, $x = \epsilon$, a very small number, the corresponding value of X is 1, i.e. the region is significantly magnified so that we can see all details of the solution as if we would use a microscope. Note that according to the chain differentiation rule

$$\frac{d}{dx} = \frac{d}{dX}\frac{dX}{dx} = \frac{1}{\epsilon}\frac{d}{dX}\,.$$

Then equation (4.8) becomes

$$\frac{du}{dX} + u = \epsilon X\,, \quad u(0) = 1\,, \quad 0 < \epsilon \ll 1\,. \tag{4.14}$$

Note that by changing an independent variable x to a stretched coordinate X we removed *singularity* from the original equation (the highest derivative term is not multiplied by a small parameter anymore) and obtained a regular perturbation problem similar to the one considered in the previous example. The solution of (4.14) is called the *inner solution*. We write

$$u_i = u_{i0} + \epsilon u_{i1} + \dots ,$$

substitute this expansion in (4.14), collect terms at different orders of ϵ and obtain

$$\epsilon^0 : \quad \frac{du_{i0}}{dX} + u_{i0} = 0 , \quad u_{i0}(0) = 1 \quad \Rightarrow \quad u_{i0} = e^{-X} , \tag{4.15}$$

$$\epsilon^1 : \quad \frac{du_{i1}}{dX} + u_{i1} = X , \quad u_{i1}(0) = 0 \quad \Rightarrow \quad u_{i1} = e^{-X} + X - 1 , \tag{4.16}$$

$$\epsilon^k : \quad \frac{du_{ik}}{dX} + u_{ik} = 0 , \quad u_{ik}(0) = 0 \quad \Rightarrow \quad u_{ik} = 0 , \ i = 2, 3, \dots \tag{4.17}$$

It follows from (4.17) that no further improvement of the inner solution is possible. This indicates that we have possibly found the exact solution using the coordinate stretching. Indeed rewriting u_i in terms of the outer coordinate x we obtain

$$u_i = e^{-\frac{x}{\epsilon}} + \epsilon \left(e^{-\frac{x}{\epsilon}} + \frac{x}{\epsilon} - 1 \right) = u \, .$$

The fact that we obtained the exact solution by stretching coordinates is a rather lucky incident. This does not usually happen for more complicated nonlinear problems where we typically have the infinite number of terms in expansions. In such problems, the inner solution is valid only for the region near the boundary and the outer solution provides a good approximation in a region away from it. In order to obtain a *composite solution* u_c which is valid throughout the whole domain, the following formula is used

$$u_c = u_i + u_o - u_{io} , \quad \text{where} \quad u_{io} = \lim_{x \to 0} u_0(x) = \lim_{X \to \infty} u_i(X) \, .$$

This is called *matching* the inner and outer solutions. *Theory of matched asymptotic expansions* is somewhat involved and we will not consider it in this course.

Finally, note that the *stretching scale* (the boundary layer thickness) for the inner coordinate is frequently chosen as $\epsilon^{\frac{1}{p}}$, where the highest derivative term entering the equation is $\epsilon \frac{d^p u}{dx^p}$. In particular, we will see in section 4.2 that the physical boundary layer thickness in the flow about a flat plate is $\sim 1/\sqrt{Re}$ because the highest (second) derivative terms in the nondimensional Navier-Stokes equations are multiplied by a small parameter $1/Re$.

Activity 4.B → Send in to the lecturer for feedback solutions for Exercises 4.1–4.3.

4.1.3 Exercises

Ex. 4.1: Expand the exact solution (4.2) into the Taylor series in ϵ (you can use symbolic software such as Maple or symbolic toolbox in matlab) and show that it coincides with (4.5). When does this series converge? Based on this, make a conclusion about the limits of validity of the asymptotic solution (4.5).

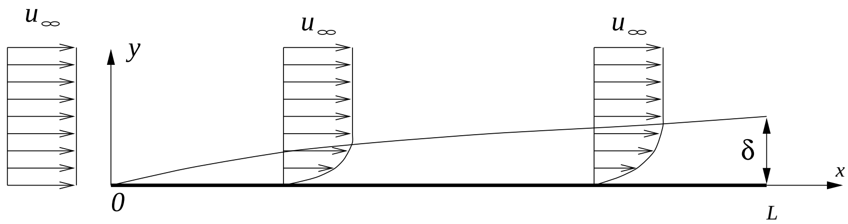


Figure 4.3: Sketch of the flow over a flat plate.

Ex. 4.2: Explain the role of each term in a composite asymptotic solution for a singular perturbation problem.

Ex. 4.3: Show that coordinate stretching $X = x/\epsilon^{1/p}$ removes singularity from the highest derivative term $\epsilon\frac{d^p u(x)}{dx^p}$ converting it to $\frac{d^p u(X)}{dX^p}$.

4.2 Boundary layer on a flat plate

In this section, we will use nondimensionalisation to deduce the relative importance of different terms in the Navier-Stokes equations describing a steady flow of viscous incompressible fluid over a thin flat plate. We will see that the equations in this case comprise a singular perturbation problem and thus techniques outlined in Section 4.1.2 can be used to find an approximate solution for such a flow.

Objectives:

- to view a set of the Navier-Stokes equations describing the flow along a flat plate as a singular perturbation problem;
- to use scaling arguments and proper nondimensionalisation in order to determine the most important terms in the equations;
- to introduce stretched coordinates in order to "zoom into" the boundary layer and to derive the Prandtl equations; and
- to derive the self-similar solution of the Blasius equation and use it to find boundary layer velocity profile and the drag force on a flat plate.

Reading 4.C → [OO95], pp. 28–38; ([Whi94], pp. 394–402; [Sch79], Chapter VII, pp. 127–131 and 133–144)

A typical steady viscous flow over a flat horizontal plate is sketched in figure 4.3. Away from the plate, the flow is uniform with velocity u_∞ in the x-direction, but the velocity is 0 on the plate surface due to the no-slip boundary condition. As a result, a relatively thin boundary layer where the longitudinal velocity u changes rapidly from $u = u_\infty$ to $u = 0$ exists at $y \to 0$ (see figure 4.4). Let us view this physical problem as a perturbation problem and try to find an asymptotic solution for the Navier-Stokes equations describing such a flow. The nondimensional Navier-Stokes equations (3.12)–(3.14) show that the highest (second) derivative terms are

29. Flat plate at zero incidence. The plate is 2 per cent thick, with beveled edges. At this Reynolds number of 10,000 based on the length of the plate, the uniform stream is only slightly disturbed by the thin laminar boundary layer and subsequent laminar wake. Their thickness is only a few per cent of the plate length, in agreement with the result from Prandtl's theory that the boundary-layer thickness varies as the square root of the Reynolds number. Visualization is by air bubbles in water. *ONERA photograph, Werlé 1974*

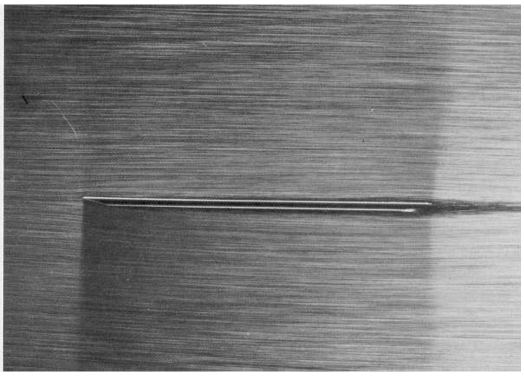


Figure 4.4: photograph from [Dyk82]

multiplied by the factor $1/Re$, where the Reynolds number $Re = \frac{U\infty L}{\nu}$ and $U = u_\infty$ in this case. $1/Re$ is a small parameter. Indeed, consider for example driving a car at 60 km/h. The kinematic viscosity of air $\nu_{air} \approx 1.5 \times 10^-5\,\mathrm{m}^2/\mathrm{s}$, the size of a bonnet is about 1.5 m. Then $1/Re \approx 6 \times 10^{-7}$! Thus the boundary layer problem on a flat plate can be treated as a singular perturbation problem as was introduced in Section 4.1.2, so in order to find an approximate solution for the flow we will need to use stretched coordinates in the direction of the highest gradients (i.e. in the direction y normal to the plate). These observations suggest that near the plate the characteristic velocity and length scales should be different for the x- and y-directions (we need to "zoom into" the boundary layer). Thus we adopt the following nondimensionalisation which is different from (3.11):

$$u' = u_\infty u\,, \quad v' = v_0 v\,, \quad x' = Lx\,, \quad y' = \delta y\,, \quad p' = \rho u_\infty^2 p\,, \tag{4.18}$$

where primes denote dimensional quantities, v_0 is the characteristic vertical velocity component and δ is the boundary layer thickness (say, the distance from the wall to the location where $u = 0.99u_\infty$) at the trailing edge $x' = L$ of the plate, both are unknown at this point. Then the continuity equation becomes

$$\frac{u_\infty}{L}\frac{\partial u}{\partial x} + \frac{v_0}{\delta}\frac{\partial v}{\partial y} = 0\,. \tag{4.19}$$

To enforce the mass conservation, both terms must be equally important to balance each other. Recollect that one of the purposes of the nondimensionalisation is to make all nondimensional quantities of order 1. This is possible only if $(u_\infty/L) \sim (v_0/\delta)$. Thus we take $v_0 = u_\infty(\delta/L)$. Then the nondimensionalised steady Navier-Stokes equations become

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{L^2}{\delta^2}\frac{\partial^2 u}{\partial y^2}\right)\,, \tag{4.20}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{L^2}{\delta^2}\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{L^2}{\delta^2}\frac{\partial^2 v}{\partial y^2}\right) - \frac{1}{Fr}\frac{L^2}{\delta^2}\,, \tag{4.21}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{4.22}$$

where $Re = \frac{u_\infty L}{\nu}$ is the Reynolds number and $Fr = \frac{u_\infty^2}{g\delta}$ is the Froud number. The ratio L/δ is expected to be very large (the boundary layer is thin). Since all nondimensional quantities are of order 1, in order to balance the term $\frac{1}{Re}\frac{L^2}{\delta^2}\frac{\partial^2 u}{\partial y^2}$ by the other terms entering (4.20) we have to have that $\frac{1}{Re}\frac{L^2}{\delta^2} \sim 1$ or $\frac{\delta}{L} = \frac{1}{\sqrt{Re}}$, the result expected from the discussion at the end of Section 4.2. Using this definition of δ we retain only the leading order terms in equations (4.20)–(4.22) and obtain a set of approximate equations valid in the boundary layer

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}\,, \tag{4.23}$$

$$\frac{\partial p}{\partial y} = 0\,, \tag{4.24}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,. \tag{4.25}$$

These equations were first derived by Prandtl and are called after him—the *Prandtl equations*. They comprise a basis of the *Prandtl boundary layer theory*. It follows from equation (4.24) that the pressure is constant across the boundary layer and thus is equal to the one in a free stream away from the plate. But away from the plate the viscous effects are negligible and flow is essentially parallel to the plate ($v = 0$). Thus the Bernoulli equation is valid in the region away from the plate

$$p' + \rho\frac{u_\infty^2}{2} + \rho g y' = \text{const.}\,,$$

or, differentiating with respect to x',

$$-\frac{\partial p'}{\partial x'} = \rho u_\infty \frac{\partial u_\infty}{\partial x'}\,.$$

Since for the flow along a flat horizontal plate $u_\infty = \text{const.}$, $\frac{\partial p'}{\partial x'} = \frac{\partial p}{\partial x} = 0$.

Let us introduce the stream function ψ such that

$$u = \frac{\partial \psi}{\partial y}\,, \quad v = -\frac{\partial \psi}{\partial x}\,.$$

Then the continuity equation is satisfied automatically and equation (4.23) becomes

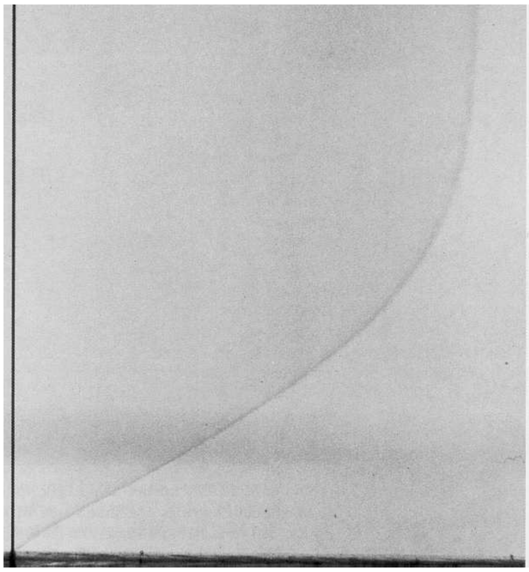
$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^3 \psi}{\partial y^3} \tag{4.26}$$

with the boundary conditions

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0 \quad \text{at} \quad x > 0\,,\ y = 0\,, \tag{4.27}$$

$$\frac{\partial \psi}{\partial y} \to 1 \quad \text{as} \quad y \to \infty\,, \tag{4.28}$$

$$\psi = 0 \quad \text{at} \quad x > 0\,,\ y = 0\,. \tag{4.29}$$



30. Blasius boundary-layer profile on a flat plate. The tangential velocity profile in the laminar boundary layer on a flat plate, discovered by Prandtl and calculated accurately by Blasius, is made visible by tellurium. Water is flowing at 9 cm/s. The Reynolds number is 500 based on distance from the leading edge, and the displacement thickness is about 5 mm. A fine tellurium wire perpendicular to the plate at the left is subjected to an electrical impulse of a few milliseconds duration. A chemical reaction produces a slender colloidal cloud, which drifts with the stream and is photographed a moment later to define the velocity profile. *Photograph by F. X. Wortmann*

Figure 4.5: photograph from [Dyk82]

Blasius, the student of Prandtl, suggested in 1908 to look for a solution of the above equation in the form

$$\psi = \sqrt{x}\, f(\eta)\,, \quad \eta = \frac{y}{\sqrt{x}}\,. \tag{4.30}$$

This solution depends on η, the specific combination of x and y. Such solutions are called *self-similar*. Substituting (4.30) in (4.26)–(4.29) we obtain the *Blasius equation* for a boundary layer on a flat plate

$$f_{\eta\eta\eta} + \frac{1}{2} f\, f_{\eta\eta} = 0\,, \quad f = f_\eta = 0 \text{ at } \eta = 0\,, \quad f_\eta \to 1 \text{ as } \eta \to \infty\,. \tag{4.31}$$

This is a nonlinear third order ordinary differential equation. Its solution cannot be given by an algebraic formula and has to be obtained numerically. It is easy to deduce that $u = f_\eta$. The corresponding velocity profile is shown in figure 2.6 ([OO95, p. 33]), $m = 0$ line. Compare it with an experimental profile shown in figure 4.5. Note the peculiar *inflection point* behaviour of the profile near the plate. This is the major difference of the *laminar* boundary layer velocity profile on a flat plate from that near the wall in a Poiseuille flow.

If we define the boundary layer thickness as a distance from the plate to the point where the longitudinal nondimensional velocity is $u = 0.99 = f_\eta(\eta_{0.99}) = \text{const.}$ then obviously $\eta_{0.99} = \text{const.}$ The numerical solution gives that $\eta_{0.99} \approx 5.0$. Then using definition of η we obtain that $y_{0.99} \approx 5.0\sqrt{x}$, i.e. the boundary layer thickness grows as a square root of the distance from the leading edge of the plate.

It is instructive to obtain an expression for the friction force acting on a plate surface (*skin friction*) in a uniform flow. By definition, the friction force per unit surface area is a viscous stress estimated on the surface of the plate

$$\begin{aligned}\sigma_{12} &= \mu\frac{u_\infty}{\delta}\frac{\partial u}{\partial y}\bigg|_{y=0} = \mu\frac{u_\infty}{L}\sqrt{\frac{Re}{x}}f_{\eta\eta}\bigg|_{\eta=0} \\ &= \rho\sqrt{\frac{\nu u_\infty^3}{xL}}f_{\eta\eta}(0) \approx 0.332\rho u_\infty\sqrt{\frac{\nu u_\infty}{x'}}\,. \end{aligned} \tag{4.32}$$

Thus the local friction force is independent of the total length of the plate and decreases as $1/\sqrt{x'}$ with the distance from the leading edge. The total *drag force on a rectangular plate* of the length L and width b then is given by an integral

$$D = b\int_0^L \sigma_{12}\,dx \approx 0.664\rho u_\infty b\sqrt{\nu u_\infty L} = \frac{0.664}{\sqrt{Re}}\rho u_\infty^2 A\,, \tag{4.33}$$

where $A = bL$ is the area of the plate. The developed Prandtl boundary layer theory is valid only for laminar attached flows. Description of *separated* or *turbulent* boundary layers arising at large Reynolds numbers requires a substantially different approach which is beyond the scope of this course.

Activity 4.D → Watch the videos "Fluid Flows", [FF85] and "Fundamentals of Boundary Layers" [FOB] ([CD08], Boundary Layer). Concentrate on boundary layers and make sure that you understand why the boundary layer near the solid wall is always a source of vorticity and why its thickness is related to how quickly the vorticity is able to diffuse from the solid wall into a free stream. Send in to the lecturer for feedback solutions for Exercises 4.4–4.8

4.2.1 Exercises

Ex. 4.4: Estimate the Froud number for the example of a car moving at 60 km/h considered in this section and show that the hydrostatic component of the pressure can be neglected in the boundary layer and the y-momentum equation is given to the leading order by (4.24). Estimate also the maximum laminar boundary layer thickness on the bonnet of the car.

Ex. 4.5: Explain the meaning of boundary conditions (4.27)–(4.29).

Ex. 4.6: Obtain the Blasius equation (4.31) by substituting (4.30) in (4.26)–(4.29).

Ex. 4.7: Show that $u = f_\eta$.

Ex. 4.8: Introduce the local Reynolds number $Re_x = \frac{u_\infty x'}{\nu}$ and the local boundary layer thickness δ_x and rewrite expression (4.32) for the skin friction on a flat plate in terms of them. Interpret the results.

4.3 Far wake

The Prandtl equations derived in Section 4.2 are applicable in any laminar flow situations where the gradients along the primary flow are much smaller than across it. Boundary layer flows about solid surfaces are not the only flows satisfying this condition. The objectives of this section are to consider two more examples where the Prandtl theory provides a powerful tool for describing the flow, namely we will derive solutions for

- a far wake behind a slender body, and
- a thin submerged jet.

Reading 4.E → [OO95], pp. 40–44; ([Sch79], Chapter IX, pp. 175–179 and 179–183)

4.3.1 Thin far wake behind a slender body

In Section 4.2, we considered a boundary layer arising on a flat plate in a uniform flow. We have shown that the velocity is zero on the plate surface because of no-slip boundary conditions, and it rapidly increases across the boundary layer to its undisturbed value u_∞. Now imagine what happens with such flow behind the trailing edge of the plate. The no-slip boundary condition is not enforced anymore and the sharp cusp-type velocity profile shown in figure 2.15 in [OO95, p. 41] starts flattening. This flattening is very quick in the immediate vicinity of the trailing edge because the inter-fluid-layer friction forces responsible for this process are proportional to $\frac{\partial u}{\partial y}$ and thus are very large. The Prandtl theory assuming slow x-variations is not applicable here. On the other hand, sufficiently far away from the trailing edge, the velocity profile differs only slightly from the uniform flow, i.e. $u = 1 - u_1$, where $u_1 \ll 1$. Term u_1 is frequently called *the defect of velocity* and the region where the defect of velocity is small is called *the far wake*. In the far wake, we again can apply the Prandtl theory to model the flow past a thin body. The actual shape and size of the body whose wake we are studying is not important at this distance!

We write equations (4.23) and (4.25) in this case in terms of u_1 (recollect that $\frac{\partial p}{\partial x} = 0$)

$$(1 - u_1)\frac{\partial u_1}{\partial x} + v\frac{\partial u_1}{\partial y} = \frac{\partial^2 u_1}{\partial y^2}\,, \tag{4.34}$$

$$\frac{\partial u_1}{\partial x} = \frac{\partial v}{\partial y}\,, \tag{4.35}$$

$$\frac{\partial u_1}{\partial y} = 0 \quad \text{at } y = 0\,, \qquad u_1 \to 0 \quad \text{when } y \to \pm\infty\,. \tag{4.36}$$

It follows from (4.35) that $v \sim u_1$ and consequently is also small. Then, to leading order, we neglect the products of small terms in (4.34) and obtain a "heat" (or diffusion) equation for u_1

$$\frac{\partial u_1}{\partial x} = \frac{\partial^2 u_1}{\partial y^2}\,. \tag{4.37}$$

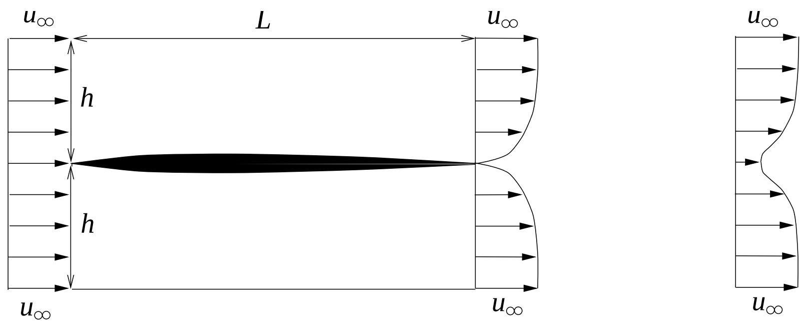


Figure 4.6: Wake velocity profile behind a slender body.

Its solution satisfying (4.36) is

$$u_1 = \frac{C}{\sqrt{x}} e^{-\frac{y^2}{4x}}, \tag{4.38}$$

where C is some arbitrary constant. We determine C using the following physical considerations.

Rewrite u_1 as a dimensional quantity recollecting the nondimensionalisation used in Section 4.2

$$u_1' = u_\infty C \sqrt{\frac{L}{x'}} e^{-\frac{u_\infty}{4\nu}\frac{y'^2}{x'}} \tag{4.39}$$

and integrate it in y' to obtain

$$\int_{-\infty}^{\infty} u_1'\, dy' = 2C\sqrt{\pi u_\infty L \nu} = K = \text{const.} \tag{4.40}$$

The result does not depend on the x'-coordinate and thus this integral is the same in the far wake and at the trailing edge of the body. What is the meaning of this invariant K? Consider the flow in a control volume with dimensions $2h \times L \times b$ as shown in figure 4.6 (b is the width of the body). Then the *kinetic energy* of the fluid entering and exiting the parallelepiped is

$$E_{k\,in} = \rho u_\infty^2 hbL$$

and

$$E_{k\,out} = \frac{1}{2}\rho bL \int_{-h}^{h} \left(u'^2 + v'^2\right) dy' \approx \rho u_\infty^2 hbL - \rho u_\infty bL \int_{-h}^{h} u_1'\, dy',$$

respectively. In these expressions we neglected the squares of u_1' and v' as they are small. Then we conclude that the fluid loses

$$\Delta E_k = E_{k\,in} - E_{k\,out} = \rho u_\infty bL \int_{-h}^{h} u_1'\, dy'$$

of its kinetic energy as it flows over the body. Now take the limit $h \to \infty$ to obtain

$$\Delta E_k = \rho u_\infty bLK.$$

The reason why fluid loses its energy is that it is decelerated by the friction on a body surface, i.e. the energy is spent to overcome the drag. Thus

$$\Delta E_k = DL\,,$$

the work done by drag along the path of length L along the body surface. Finally, using (4.40) we determine constant C from

$$DL = \rho u_\infty bLK = 2\rho u_\infty bLC\sqrt{\pi u_\infty L\nu}$$

which results in

$$C = \frac{D}{2\rho u_\infty b\sqrt{\pi u_\infty L\nu}}\,.$$

Thus the dimensional velocity profile in a far wake is given by

$$u' = u_\infty - \frac{D}{2\rho b\sqrt{\pi u_\infty \nu x'}}e^{-\frac{u_\infty}{4\nu}\frac{y'^2}{x'}}\,. \tag{4.41}$$

Using expression (4.33) for the drag force acting on a single surface of a flat plate of the length L we obtain for the wake behind a plate

$$u' \approx u_\infty\left(1 - 0.664\sqrt{\frac{L}{\pi x'}}e^{-\frac{u_\infty}{4\nu}\frac{y'^2}{x'}}\right)\,.$$

This expression shows that the maximum deviation of the wake velocity from the free stream velocity occurs behind the body at $y' = 0$ and decays as one over the square root of the distance from the trailing edge. Thus the minimum velocity in a wake is

$$u'_m \approx u_\infty\left(1 - 0.664\sqrt{\frac{L}{\pi x'}}\right)\,.$$

Activity 4.F → Send in to the lecturer for feedback solutions for Exercises 4.9, 4.10.

4.3.2 Exercises

Ex. 4.9: Use nondimensionalisation (4.18) and relation between different scales derived in Section 4.2 to obtain (4.39) from (4.38).

Ex. 4.10: Estimate the distance behind the flat plate of the length $L = 1\,\mathrm{m}$ where the maximum deviation of the wake velocity from the free stream value does not exceed 5%. Estimate the characteristic wake thickness at that location as the distance between the points located on either side of the wake symmetry line where the deviation of the velocity from u_∞ is 5% of the maximum velocity defect $u_\infty - u'_m$. Estimate the characteristic dimensional velocity gradients in x- and y-directions and based on their ratio conclude on validity of the Prandtl theory for describing far wakes.

4.4 Summary

- Problems where a small parameter multiplies terms other than the highest order derivatives are called regular perturbation problems. Their leading order approximate solutions are obtained by simply neglecting terms multiplied by a small parameter with subsequent improvement by including neglected terms at a later stage.

- Problems where a small parameter multiplies the highest order derivatives are called singular perturbation problems. They always result in a boundary layer type solutions. Their approximate solutions are obtained by combining the outer solution, which is valid in the region away from the boundary, and the inner solution, which is valid in the boundary layer. The inner solution is usually obtained in terms of the stretched coordinate $X = \frac{x}{\epsilon^{1/p}}$, where the highest derivative term entering the equation is $\epsilon \frac{d^p u}{dx^p}$, $0 < \epsilon \ll 1$.

- The Navier-Stokes equations describing the boundary layer on a flat horizontal plate comprise a singular perturbation problem which we treat using stretched y-coordinate. The Prandtl equations (4.23)–(4.25) are leading approximation for the complete Navier-Stokes equations in the boundary layer. In turn, they give rise to the Blasius equation (4.31) on a flat plate. This is a nonlinear ordinary differential equation which is solved numerically. The numerical solution is then used to obtain the coefficients in expressions for the boundary layer thickness and the drag force on a plate. The boundary layer thickness grows as a square root of the distance from the leading edge while the local friction force per unit area is inversely proportional to this square root.

- In the far wake behind a slender body, the longitudinal velocity profile is approximately given by (4.41). The defect of velocity is proportional to the drag force acting on a body whose wake is under consideration. It decays exponentially quickly in the direction perpendicular to the free stream and as one over the square root of the distance from the trailing end of the body in the direction of the free stream.

Module 5 Slow Flows

Module contents

5.1 Low Reynolds number flows

Objectives:

- consider flows characterised by small Reynolds numbers;
- learn when such flows occur;
- see that these flows are driven by large pressure gradients.

Reading 5.A → [OO95], p. 50; ([Sch79], Chapter VI, pp. 112–113)

Recollect the definition of the Reynolds number $Re = \frac{UL}{\nu}$. Flow situations which are characterised by the small Reynolds numbers occur when one or more of the following conditions are satisfied:

- very viscous flow (large ν as in honey, syrup)
- very slow flow (e.g. capillary flows in fine blood vessels)
- motion of fine particles in fluids (microscopic rising bubbles, dust sedimentation, mist droplets, suspensions and mixtures)

Standard nondimensionalisation (3.11) leads to a system of equations (3.12)–(3.14). Multiply the first two of these equations by Re which is small. Then the highest derivative terms are of order 1 and as a result we are dealing with a regular perturbation problem. No coordinate stretching is necessary[1].

In the limit $Re \to 0$, we would obtain

$$\nabla^2 \mathbf{v} = 0, \quad \nabla \cdot \mathbf{v} = 0\,.$$

[1] This is not completely true. As seen from a simple example in [OO95, pp. 51–52] an "exotic" boundary layer at infinity might exist where compression rather than stretching of coordinates is necessary, but this is not the case in the examples we will consider here.

Figure 5.1: Very slow flow.

In two dimensions, this is a system of three equations for the two velocity components—the so-called *over-defined system*. In general such a system does not have a solution. This mathematical problem is a manifestation of the fact that such a model does not take into account some important physics of the flow. Indeed, what is the force which could maintain flow in all situations listed above? This cannot be inertia as the velocities are very small. This cannot be gravity as it can drive the fluid in vertical pipes only. This cannot be friction as in general it slows the fluid down by dissipating its kinetic energy. The only possibility is the pressure gradient. Thus the pressure has to balance the viscous terms in the equations, i.e.

$$\nabla p' \sim \mu \nabla^2 \mathbf{v}' ,$$

and thus the pressure scale is $p_0 = \frac{\mu U}{L}$. Then retaining only the largest terms in the derived nondimensional Navier-Stokes equations we obtain

$$\nabla p = \nabla^2 \mathbf{v} , \tag{5.1}$$

$$\nabla \cdot \mathbf{v} = 0 . \tag{5.2}$$

These equations describe a very slow *creeping* flow. Taking divergence of equation (5.1) we obtain

$$\nabla^2 p = 0 , \tag{5.3}$$

i.e. pressure is a *harmonic* function for creeping flows.

Activity 5.B → Watch the first part of the video "Low-Reynolds-Number Flow" [LRN] ([CD08], Low Reynolds Number). Pay attention to the common characteristics of different low Reynolds number flows.

Note the actual values of the Reynolds number in these examples. Send in to the lecturer for feedback solutions for Exercises 5.1 and 5.2.

5.1.1 Exercises

Ex. 5.1: Check that combination $\mu U/L$ has units of pressure.

Ex. 5.2: Show how equation (5.3) is obtained.

5.2 Lubrication theory

In Section 5.1 we studied very slow flows arising in viscous fluids. Such flows are characterised by $Re \to 0$ and therefore by negligible inertia. It is interesting that the mathematical approach which we used to derive approximate equations describing such flows is applicable for flows with the finite Reynolds number, but arising in very narrow gaps or slits in various machinery with parts moving relatively to each other.

Objectives:

- to develop an approximate lubrication theory of flows in narrow gaps of arbitrary shape;
- to understand how a thin film of oil can support enormous loads in bearings; and
- to model a squeeze flow between two close surfaces.

Reading 5.C → [OO95], p. 64–71; ([Sch79], Chapter VI, pp. 116–123)

5.2.1 Slider bearing theory

Consider the flow of viscous incompressible fluid in the gap of variable height caused by the motion of the lower wall as shown in figure 4.1 in [OO95, p. 65]. In a two dimensional case such a flow is similar to the Couette flow considered in Section 3.5.1, but the complication arises from the fact that the plates are non-parallel now. These flows arise in such important practical application as, for example, *slider bearings*. We will derive an approximate solution for the case when the characteristic gap height δ is small[2] compared to the bearing width L. The two-dimensional steady Navier-Stokes equations describing the flow are

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{1}{\rho}\frac{\partial P'}{\partial x'} + \nu\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right), \tag{5.4}$$

$$u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -\frac{1}{\rho}\frac{\partial P'}{\partial y'} + \nu\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right), \tag{5.5}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \tag{5.6}$$

[2] In contrast to [OO95], here we denote by δ the dimensional gap height.

where primes denote dimensional quantities. Note that in the case of incompressible fluid flowing in a uniform gravitational field $\mathbf{g} = (g_x, g_y)$ we can absorb the hydrostatic term by introducing the *modified pressure* $P' = p' - \rho(g_x x + g_y y)$ so that it does not appear explicitly in the momentum equations. The boundary conditions for the problem are those of no-slip for velocities $u' = v' = 0$ at $y' = \delta H(x)$, where $H(x)$ defines the shape of a bearing. In addition we assume that the bearing ends are open to the ambient and thus the pressure p' is equal to the atmospheric pressure p_{atm} at $x' = 0$ and $x' = L$. It is convenient to take $p_{\text{atm}} = 0$, i.e. to measure the pressure as the deviation from the ambient one. The natural nondimensionalisation is

$$x' = Lx, \quad y' = \delta y, \quad u' = Uu, \quad v' = Vv, \quad P' = P_0 P\,, \tag{5.7}$$

where, as follows from the continuity equation, $V = U\delta/L$. Under this nondimensionalisation the Navier-Stokes equations become

$$Re\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{P_0 L}{\mu U}\frac{\partial P}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{L^2}{\delta^2}\frac{\partial^2 u}{\partial y^2}\,, \tag{5.8}$$

$$Re\frac{\delta}{L}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{P_0 L^3}{\mu U \delta^2}\frac{\partial P}{\partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{L^2}{\delta^2}\frac{\partial^2 v}{\partial y^2}\,, \tag{5.9}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,. \tag{5.10}$$

Oil used in bearings has a typical density $\rho \sim 900\,\text{kg/m}^3$ and a dynamic viscosity $\mu \sim 10^{-1}\,\text{kg/(m}\cdot\text{s)}$ (see, for example, Appendix A in [Whi94] for the table of fluid properties). The bearing sizes and velocities of moving parts vary widely. Assume that $L \sim 0.1\,\text{m}$, $\delta \sim 1\,\text{mm}$ and $U \sim 1\,\text{m/s}$. Then the Reynolds number $Re = \frac{\rho U L}{\mu} \sim 900$ while $\frac{\delta}{L} \sim 10^{-2}$. Thus to leading order the inertia terms in the momentum equations (the left-hand sides in (5.8) and (5.9)) can be neglected similarly to slow flow situations considered in Section 5.1. The balance in equation (5.8) is possible only if

$$\frac{P_0 L}{\mu U} = \frac{L^2}{\delta^2} \quad \text{or} \quad P_0 = \frac{\mu U L}{\delta^2}\,.$$

Note that for the physical parameters described above the characteristic modified pressure $P_0 \sim 10^4\,\text{Pa}$ whereas its hydrostatic component $p_h = \rho g \delta \sim 9\,\text{Pa}$ and thus could be neglected from the very beginning. Then retaining only the largest terms in equations (5.8)–(5.10) we arrive at

$$\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2}\,, \quad \frac{\partial p}{\partial y} = 0\,. \tag{5.11}$$

From the second equation it follows that $p = p(x)$. Then integrating the first equation twice and applying the boundary conditions we obtain

$$u(x,y) = \underbrace{1 - \frac{y}{H}}_{\text{Couette}} + \underbrace{\frac{1}{2}\frac{\partial p}{\partial x}y(y-H)}_{\text{Poiseuille}}\,. \tag{5.12}$$

Recognise that the horizontal velocity distribution is essentially given by two familiar components (see Sections 3.5.1 and 3.5.4): the Couette flow (due to the motion of a lower wall) and the Poiseuille flow (due to the induced pressure gradient). Since $v(x, 0) = 0$ we can integrate the continuity equation to obtain

$$v(x,y) = \int_0^y \frac{\partial v(x,Y)}{\partial Y}\,dY = -\int_0^y \frac{\partial u(x,Y)}{\partial x}\,dY \tag{5.13}$$

and in particular

$$v(x,H) = 0 = -\int_0^H \frac{\partial u}{\partial x}\,dy\,.$$

According to the *Leibnitz rule* for differentiation of integrals with variable limits

$$\int_0^H \frac{\partial u}{\partial x}\,dy = \frac{d}{dx}\int_0^H u\,dy - \frac{dH}{dx}\underbrace{u(x,H(x))}_{0}\,.$$

Thus $\frac{d}{dx}\int_0^H u\,dy = 0$ or using (5.12)

$$2\int_0^H u\,dy = -\frac{H^3}{6}\frac{\partial p}{\partial x} + H = H_0 = \text{const.} \tag{5.14}$$

Finally, since $p(0) = p(1) = 0$ we obtain

$$p(x) = 6\int_0^x \frac{H(X)-H_0}{H^3(X)}\,dX\,, \quad \text{where} \quad \int_0^1 \frac{H(X)-H_0}{H^3(X)}\,dX = 0\,. \tag{5.15}$$

Note that it follows from (5.14) that $\frac{\partial p}{\partial x} = 0$ when $H = H_0$. Thus the physical meaning of constant H_0 is that the pressure in a bearing reaches its maximum or minimum at the point where $H = H_0$.

Consider as an example a linear bearing gap $H = 1 + ax$, where $a > -1$. Formula (5.15) then gives

$$p = \frac{6ax}{2+a}\,\frac{x-1}{(1+ax)^2}\,. \tag{5.16}$$

Pressure distribution in linear converging ($a = -1/4$) and diverging ($a = 1/4$ and $a = 2$) bearings is shown in figure 5.2. Note that for divergent plates the pressure in a bearing is negative, i.e. below atmospheric. The danger of this situations is that lubricant can *cavitate* i.e. evaporate intensively without high temperature boiling. Cavitation vapour bubbles are unstable and shrink quickly as soon as pressure increases. This is equivalent to mini-explosions (remember what happens when water drops to hot oil in a fry-pan) and leads to intensive wear of mechanisms such as journal bearings considered in [OO95, p. 69].

Now let us calculate force $d\mathbf{F} = (dF_x, dF_y)$ acting on a unit surface area of a linear bearing. The slope of the upper surface of a bearing is

$$\tan\theta = \frac{\delta}{L}\frac{dH}{dx} = a\frac{\delta}{L}$$

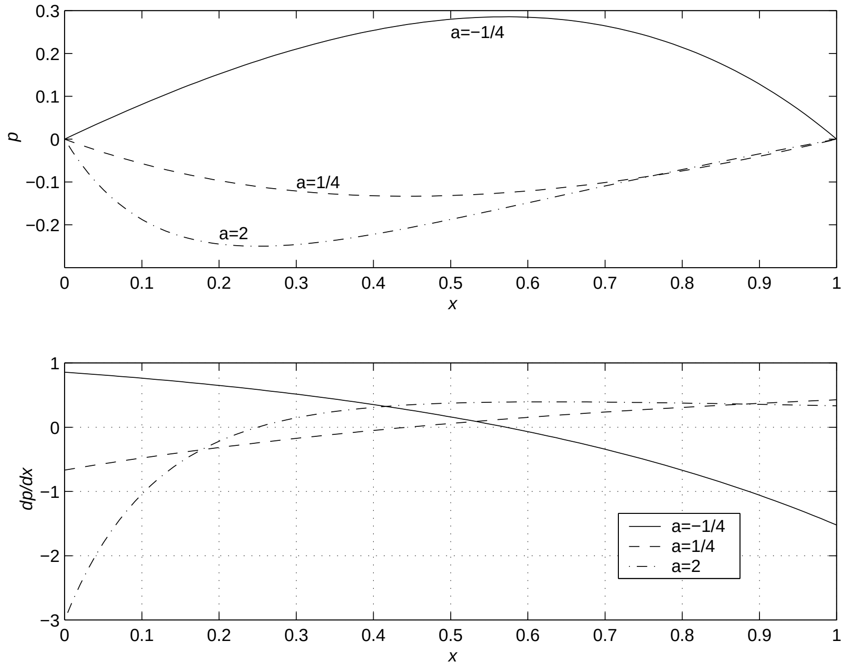


Figure 5.2: Pressure distribution along a slider bearing.

and consequently the normal to the upper surface pointing into fluid is $\mathbf{n} = (n_x, n_y) = (\sin\theta, -\cos\theta) \approx (a\frac{\delta}{L}, -1)$, where $\theta \approx a\delta/L \ll 1$. The stresses on the upper surface (see Section 3.2) are

$$\begin{aligned}
\begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{yx} & \sigma'_{yy} \end{bmatrix} &= \begin{bmatrix} -p' + 2\mu\frac{\partial u'}{\partial x'} & \mu\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right) \\ \mu\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right) & -p' + 2\mu\frac{\partial v'}{\partial y'} \end{bmatrix} \\
&= \frac{\mu U L}{\delta^2}\begin{bmatrix} -p + 2\frac{\delta^2}{L^2}\frac{\partial u}{\partial x} & \frac{\delta}{L}\left(\frac{\partial u}{\partial y} + \frac{\delta^2}{L^2}\frac{\partial v}{\partial x}\right) \\ \frac{\delta}{L}\left(\frac{\partial u}{\partial y} + \frac{\delta^2}{L^2}\frac{\partial v}{\partial x}\right) & -p + 2\frac{\delta^2}{L^2}\frac{\partial v}{\partial y} \end{bmatrix} \\
&\approx \frac{\mu U L}{\delta^2}\begin{bmatrix} -p & \frac{\delta}{L}\frac{\partial u}{\partial y} \\ \frac{\delta}{L}\frac{\partial u}{\partial y} & -p \end{bmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
dF_x &= \sigma_{xx}n_x + \sigma_{xy}n_y \approx -\frac{\mu U L}{\delta^2}\left(\frac{ap\delta}{L} + \frac{\delta}{L}\frac{\partial u}{\partial y}\right) = -\frac{\mu U}{\delta}\left(ap + \frac{\partial u}{\partial y}\right), \\
dF_y &= \sigma_{yx}n_x + \sigma_{yy}n_y \approx \frac{\mu U L}{\delta^2}p.
\end{aligned}$$

The total forces F_x and F_y acting on a linear slider bearing of width b are computed by integrating the above results over the bearing length

$$F_x = b\frac{\mu UL}{\delta}\int_0^1 \left(ap + \frac{\partial u}{\partial y}\right) dx = 2b\frac{\mu UL}{\delta}\,\frac{5(2+a)\ln(1+a) - 9a}{a(2+a)} \tag{5.17}$$

$$F_y = b\frac{\mu UL^2}{\delta^2}\int_0^1 p\,dx = 6b\frac{\mu UL^2}{\delta^2}\,\frac{2a - (2+a)\ln(1+a)}{a^2(2+a)} \tag{5.18}$$

so that $\frac{F_x}{F_y} \sim \frac{\delta}{L}$. Thus the drag force F_x acting on the upper surface is much smaller than the load F_y which can be supported by a bearing. Note that the "dry" friction force which would occur between two solid surfaces sliding with respect to each other without any liquid lubricant is many orders of magnitude larger, namely, $\frac{F_x}{F_y} \sim 0.2 - 1.2$ depending on the surface material and roughness. This explains the importance of liquid lubrication and why bearings are used so widely in practice.

Activity 5.D → Watch the video "Low Reynolds Number Flow" [LRN] ([CD08], Hydrodynamic Lubricationm). Concentrate on the part on hydrodynamic lubrication. See how important a thin fluid film between two moving parts is for reducing the friction. Send in to the lecturer for feedback solutions for Exercises 5.3–5.6.

5.2.2 Exercises

Ex. 5.3: Why a must be greater than -1 for a linear slider bearing with the upper surface given nondimensionally by $H = 1 + ax$? Obtain expression (5.16) for the pressure distribution in such a bearing.

Ex. 5.4: Obtain nondimensional velocities for a linear slider bearing with $H = 1 + ax$. Plot velocity fields for $a = -1/4$, $1/4$ and 2. Interpret the results.

Ex. 5.5: Explain why high speed water skiing is possible even if the ski material is heavier than water.

Ex. 5.6: What are the forces acting on the upper wall of a linear bearing when $a = 0$? Hint: use the Taylor series expansions of formulae (5.17) and (5.18) as $a \to 0$. You may do it by hand or with the help of some symbolic software, e.g. Maple, Mathematica, Reduce or symbolic toolbox in matlab). Alternatively you can use the L'Hôpital rule. Compare your answers with the results obtained in Section 3.5.1 for the plane Couette flow.

5.2.3 Squeeze films

So far we derived expressions for flow quantities in gaps of fixed shape. We also see from (5.16) that the pressure between two parallel walls (when $a = 0$) is zero and thus no external force on the gap walls in this case could be supported (see **Ex. 5.6.**): any force normal to the walls would lead to decreasing gap height δ. The fluid (lubricant) would eventually be forced out from the gap and the dry friction regime would be established. So is there any point in lubricating parallel moving parts in machinery?

Consider the flow between two surfaces as shown in figure 4.8 in [OO95, p. 70]. Let the lower wall be a resting flat plate while the upper wall (which may be deformed) is moving from or towards it. Its velocity at $t' = 0$ is $V_0'(x') = V_0 v_0(x)$ and the characteristic gap height is $\delta \ll L$ such that the initial dimensional gap shape is given by $H'(x', 0) = \delta H_0(x)$, where $H_0 = \mathcal{O}(1)$ and $v_0 = \mathcal{O}(1)$. The flow is unsteady and the dimensional Navier-Stokes equations are

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{1}{\rho}\frac{\partial P'}{\partial x'} + \nu\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right), \tag{5.19}$$

$$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -\frac{1}{\rho}\frac{\partial P'}{\partial y'} + \nu\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right), \tag{5.20}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{5.21}$$

with boundary conditions

$$u' = v' = 0 \quad \text{at} \quad y' = 0 \quad \text{and} \quad u' = 0,\ v' = \frac{\partial H'}{\partial t'} \quad \text{at} \quad y' = H'(x', t')$$

and initial conditions

$$v'(x', H'(x', 0), 0) = V_0 v_0(x), \quad H'(x', 0) = \delta H_0(x)\,.$$

We nondimensionalise the above equations using

$$x' = Lx, \quad y' = \delta y, \quad u' = V_0\frac{L}{\delta}u, \quad v' = V_0 v, \quad P' = \frac{\mu V_0 L^2}{\delta^3}P, \quad t' = \frac{\delta}{V_0}t \tag{5.22}$$

and after retaining only the largest terms we obtain a system of approximate equations (5.9)–(5.10) as before with the following conditions

$$u = v = 0 \quad \text{at} \quad y = 0\,; \qquad u = 0,\ v = \frac{\partial H}{\partial t} \quad \text{at} \quad y = H(x, t)\,;$$
$$v(x, H(x, 0), 0) = v_0(x), \quad H(x, 0) = H_0(x)\,.$$

In addition we assume that the gap is open to atmosphere so that $P(0, t) = P(1, t) = 0$. Similarly to Section 5.2.1, when the gap height δ is very small, the hydrostatic component of the modified pressure P can be neglected so that below we will use $p \approx P$. Since the problem is reduced to the one considered in Section 5.2.1 the velocity field is given by equations (5.12) and (5.13). Then

$$v(x, H(x, t)) = \frac{\partial H}{\partial t} = -\frac{\partial}{\partial x}\int_0^H u\,dy$$

Integrating (5.12) we obtain the model for a slow squeeze flow

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(\frac{H^3}{12}\frac{\partial p}{\partial x}\right) \tag{5.23}$$

If the upper surface is also a flat solid plate then $H = H(t)$ and (5.23) leads to

$$\frac{12}{H^3}\frac{dH}{dt} = \frac{\partial^2 p}{\partial x^2}\,.$$

Since the left-hand side of the above equation depends only on t then

$$12\frac{dH}{dt} = C(t)H^3 \quad \text{and} \quad \frac{\partial^2 p}{\partial x^2} = C(t)\,.$$

Therefore $p = \frac{C(t)}{2}x^2 + C_1(t)x + C_2(t)$ or, applying the boundary conditions, $p = \frac{C(t)}{2}x(x-1)$. In order to determine function $C(t)$ we have to use an additional condition. Note that the upper plate does not immediately fall on the lower one under the action of the external load F_y only because it takes some time to squeeze all lubricant out from the gap. If the upper plate moves slowly then we can neglect its acceleration or deceleration and at any particular time $\int_0^1 p\,dx \approx F_y$. Then if F_y is constant, the pressure integral must not depend on time. Then $C(t) = \text{const}$. In our nondimensionalisation $H_0 = 1$ and $v_0 = -1$ for a flat upper plate, thus $C(t) = -1$. Simple integration then leads to

$$H = \frac{1}{\sqrt{2t+1}} \quad \text{and} \quad p = 6x(1-x)\,. \tag{5.24}$$

This shows that it would take infinite time to squeeze all lubricant out from the gap between two parallel plates! Thus although the lubricant in this case cannot support any external load its thin layer will always be present between the plates and regime of dry friction will never occur.[3] It explains why moving parts of any mechanism are always being lubricated regardless of their geometry.

Activity 5.E → Compare nondimensionalisations (5.7) and (5.22). Note and explain the differences, send in your conclusions to the lecturer for feedback.

5.3 Summary

- In very slow (creeping) Stokes flows of incompressible fluid the viscous forces are balanced by large pressure gradients or buoyancy forces. Inertia is not important. Equations (5.1)–(5.2) describe such flows.
- The mathematical approach used to describe creeping flows with $Re \to 0$ can be successfully applied to flows caused by the Couette type wall motions at moderate Reynolds numbers in narrow gaps and slits. Such flows typically arise in bearings and other machinery with lubricated sliding parts. The leading approximation of the flow fields then is completely defined by the shape of the gap through formulae (5.12), (5.13) and (5.15). The pressure in convergent gaps is always higher than atmospheric and thus can support an external load. The pressure in divergent gaps is lower than ambient and this might cause cavitation of a lubricant.

[3] Here we have assumed that the surfaces are ideal. In reality the smallest possible gap height is determined by the surface roughness δ_r. It would take finite time $t_r = \frac{\delta}{2V_0}\left(\frac{\delta^2}{\delta_r^2} - 1\right)$ to reduce the gap height to that level.

- Load applied to parallel lubricated walls squeezes the fluid out of the gap, but it would take infinite time to do it completely (equation (5.24)). Thus lubrication makes sense even when the gap between the moving parts has parallel walls. The distribution of pressure and velocities in squeeze flow are given by equations (5.23) and (5.12) and (5.13), respectively.

Module A Appendix

Equations of Vector Algebra

In the following identities $\mathbf{a}$ and $\mathbf{b}$ represent vector and ϕ and ψ scalar functions. Symbol $\boldsymbol{\nabla}$ denotes the gradient vector, $\boldsymbol{\nabla}\cdot$ is the divergence, $\boldsymbol{\nabla}\times$ is the curl and ∇^2 is the Laplacian. The priorities which determine the order in which operations are performed are:

1. Vector product $\times$;
2. Scalar product $\cdot$;
3. Silent product (no symbol);
4. Summation $+$ and subtraction $-$.

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} \tag{A.1}$$
$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} \tag{A.2}$$
$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} \tag{A.3}$$
$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} \tag{A.4}$$
$$\mathbf{a}\times\mathbf{a} = \mathbf{0} \tag{A.5}$$
$$[\mathbf{abc}] \equiv \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c} \tag{A.6}$$
$$[\mathbf{abc}] = [\mathbf{bca}] = [\mathbf{cab}] = -[\mathbf{acb}] = -[\mathbf{bac}] = -[\mathbf{cba}] \tag{A.7}$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \tag{A.8}$$
$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a} \tag{A.9}$$
$$\boldsymbol{\nabla}(\phi+\psi) = \boldsymbol{\nabla}\phi+\boldsymbol{\nabla}\psi \tag{A.10}$$
$$\boldsymbol{\nabla}(\phi\psi) = \psi\boldsymbol{\nabla}\phi+\phi\boldsymbol{\nabla}\psi \tag{A.11}$$
$$\boldsymbol{\nabla}\cdot(\mathbf{a}+\mathbf{b}) = \boldsymbol{\nabla}\cdot\mathbf{a}+\boldsymbol{\nabla}\cdot\mathbf{b} \tag{A.12}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi = \nabla^2\phi \tag{A.13}$$
$$\boldsymbol{\nabla}\times(\mathbf{a}+\mathbf{b}) = \boldsymbol{\nabla}\times\mathbf{a}+\boldsymbol{\nabla}\times\mathbf{b} \tag{A.14}$$
$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi = \mathbf{0} \tag{A.15}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{a} = \mathbf{0} \tag{A.16}$$
$$\mathbf{a}\times\boldsymbol{\nabla}\cdot\mathbf{b} = \mathbf{a}\cdot\boldsymbol{\nabla}\times\mathbf{b} \tag{A.17}$$
$$\boldsymbol{\nabla}\cdot(\phi\mathbf{a}) = \phi\boldsymbol{\nabla}\cdot\mathbf{a}+\mathbf{a}\cdot\boldsymbol{\nabla}\phi \tag{A.18}$$
$$\boldsymbol{\nabla}\times(\phi\mathbf{a}) = \phi\boldsymbol{\nabla}\times\mathbf{a}+\boldsymbol{\nabla}\phi\times\mathbf{a} \tag{A.19}$$
$$\boldsymbol{\nabla}\cdot\mathbf{a}\times\mathbf{b} = \mathbf{b}\cdot\boldsymbol{\nabla}\times\mathbf{a}-\mathbf{a}\cdot\boldsymbol{\nabla}\times\mathbf{b} \tag{A.20}$$
$$\begin{aligned}\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\ \boldsymbol{\nabla}\cdot\mathbf{b}-\mathbf{b}\ \boldsymbol{\nabla}\cdot\mathbf{a}+\mathbf{b}\cdot\boldsymbol{\nabla}\ \mathbf{a}\\ -\mathbf{a}\cdot\boldsymbol{\nabla}\ \mathbf{b}\end{aligned} \tag{A.21}$$

$$\begin{aligned}\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{b}) + \mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{a})\\ +(\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a}\end{aligned} \quad \text{(A.22)}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{a}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{a}) - \nabla^2\mathbf{a} \quad \text{(A.23)}$$

$$\nabla^2\mathbf{a} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{a}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{a}) \quad \text{(A.24)}$$

$$\boldsymbol{\nabla}\left(\frac{\phi}{\psi}\right) = \frac{\psi\boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\psi}{\psi^2} \quad \text{(A.25)}$$

Index