

Figure 1: domain and coarse grid.

Question 1.

(a) Consider the problem

$$\nabla^2 u + xyu = xy(6 + x^3y)$$

on the irregular domain shown in Figure 1. The boundary conditions are

$$u(0, y) = 0, \quad u(x, 2) = 2x^3, \quad u_y(x, 0) = x^3$$

and

$$u = x^3(4 - 2x)$$

along the inclined right boundary. Consider finding a numerical solution on the coarse grid shown in the figure, and adopt the shown lexicographic order of points.

- i. Use the boundary conditions to specify the boundary values for u at the boundary grid points where possible. 1 marks
- ii. Apply the finite difference approximation for the equation at the interior point on a non-uniform grid. State what the accuracy of your scheme is. Show that this approximation leads to the equation
$$5u_5 - u_2 = 4.$$
 4 marks
- iii. Approximate the lower boundary condition 4 marks
 - A. using fictitious points approach, and
 - B. using a first order forward difference.
- iv. Solve the systems of two linear equations arising from (ii) & (iii.A) and from (ii) & (iii.B). Compare the result with the exact solution $u = x^3y$. Which scheme produces more accurate results? Give a brief justification of your answer. 4 marks
- v. Rewrite the system from (ii) & (iii.A), corresponding to the fictitious points approach, in the form suitable for Jacobi iteration. 2 marks
- vi. Perform two Jacobi iterations for this system starting from an initial guess of zero. 2 marks

- vii. Use the results of (v) to write the corresponding equations for weighted Jacobi iteration with weight ω . 2 marks

(b) Consider the partial differential equation

$$u_t = \nabla^2 u + xyu.$$

- i. Use the Crank-Nicolson method to approximate this equation. 4 marks
- ii. Is this scheme explicit or implicit? Briefly state what the accuracy of the scheme is. 2 marks

Question 2.

(a) Consider the 6 state Markov chain with the following transition matrix: 7 marks

$$P = \begin{bmatrix} 4/9 & 0 & 4/9 & 1/9 & 0 & 0 \\ 0 & 2/3 & 1/3 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- i. Draw a corresponding state transition diagram.
- ii. Identify any closed sets, and any transient, recurrent or absorbing states. Give reasons.
- iii. Is the Markov chain ergodic? Give your reason.
- (b) At a web publishing firm, page designers need to use a scanner in order to help create web pages. The firm operates 10 hours a day, 5 days a week, and the 20 designers need to use the scanner on average once a day. The designers arrive at the scanner at random according to a Poisson process, and the time a designer needs to use the scanner is exponentially distributed with an average of 20 minutes. 18 marks

The firm's designers frequently complain that the waits experienced at the scanner are too long, often waiting longer than an hour, and that productivity is being lost. On the other hand, management contends that the scanner is only being used for two-thirds of the time, and thus buying another scanner is wasteful.

Use queuing theory to analyse the situation (ignore the breaks between the end of one working day and the start of the next, and assume that 20 designers is large enough to be effectively an infinite number). You may want to refer to Tables 1 and 2 to help answer parts of this question.

- i. Is management correct in saying the scanner is being used only two-thirds of the time? Give reasons.
- ii. What fraction of the times that a designer needs to use the scanner will he/she have to wait an hour or longer before using it?

Table 1: $P\{j \geq s\}$ for the $M/M/s/GD/\infty/\infty$ queuing system

ρ	$s = 2$	$s = 3$	$s = 4$	$s = 5$	$s = 6$	$s = 7$	$s = 8$	$s = 9$	$s = 10$
0.05	0.005	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.10	0.018	0.004	0.001	0.000	0.000	0.000	0.000	0.000	0.000
0.15	0.039	0.011	0.003	0.001	0.000	0.000	0.000	0.000	0.000
0.20	0.067	0.025	0.010	0.004	0.002	0.001	0.000	0.000	0.000
0.25	0.100	0.044	0.020	0.010	0.005	0.002	0.001	0.001	0.000
0.30	0.138	0.070	0.037	0.020	0.011	0.006	0.004	0.002	0.001
0.35	0.181	0.102	0.060	0.036	0.022	0.014	0.009	0.006	0.004
0.40	0.229	0.141	0.091	0.060	0.040	0.027	0.018	0.013	0.009
0.45	0.279	0.186	0.129	0.091	0.065	0.047	0.035	0.026	0.019
0.50	0.333	0.237	0.174	0.130	0.099	0.076	0.059	0.046	0.036
0.55	0.390	0.293	0.227	0.179	0.143	0.115	0.094	0.076	0.063
0.60	0.450	0.355	0.287	0.236	0.197	0.165	0.140	0.119	0.101
0.65	0.512	0.421	0.354	0.303	0.261	0.227	0.198	0.174	0.154
0.70	0.576	0.492	0.429	0.378	0.336	0.301	0.271	0.245	0.222
0.75	0.643	0.568	0.509	0.462	0.422	0.387	0.357	0.330	0.307
0.80	0.711	0.647	0.596	0.554	0.518	0.486	0.458	0.432	0.409
0.85	0.781	0.730	0.689	0.654	0.624	0.597	0.573	0.550	0.530
0.90	0.853	0.817	0.788	0.762	0.740	0.720	0.702	0.685	0.669
0.95	0.926	0.907	0.891	0.878	0.866	0.854	0.844	0.835	0.826

- iii. Suppose the firm proposes to buy a second scanner to operate in parallel to the first. For both situations determine the times a designer would expect to waste waiting. Given that a web designer costs the firm \$25 per hour in wages and on-costs, and that a quality scanner costs \$5,200 per year in depreciation, maintenance and support, is the decrease in expected waiting times worth the cost of the second scanner?

Question 3.

- (a) Consider a set of polynomials $1, x, x^2$ on the interval $0 \leq x \leq 1$ with the inner product $\langle f, g \rangle = \int_0^1 fg dx$.
- Show by applying to $\{1, x, x^2\}$ the Gram-Schmidt procedure that the orthonormal basis for the vector space of polynomials of degree not exceeding three on this interval can be chosen as $\{1, \sqrt{3}(2x - 1), \sqrt{5}(6x^2 - 6x + 1)\}$. 4 marks
 - Find the projection u_p of the function $u = x^3$ onto this vector space using the above basis and the Projection Theorem. 4 marks
- (b) Use the Euclidean norm to show that multiplication of a two-component vector $\mathbf{x} = (x_1, x_2)$ in \mathbf{R}^2 by a rotation matrix 3 marks

$$\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

Table 2: summary of formulae for three common queuing systems

	M=M=1=GD=1=1	M=M=1=GD=c=1	M=M=s=GD=1=1
π_0	$1 - \rho$	$\frac{1-\rho}{1-\rho^{c+1}}$	$\left[\sum_{i=0}^{s-1} \frac{(s\rho)^i}{i!} + \frac{(s\rho)^s}{s!(1-\rho)} \right]^{-1}$
π_j	$\rho^j(1-\rho)$	$\begin{cases} \rho^j \pi_0 & j = 0, \dots, c \\ 0 & j = c+1, \dots \end{cases}$	$\begin{cases} \frac{(s\rho)^j}{j!} \pi_0 & j = 0, \dots, s \\ \frac{(s\rho)^j}{s!s^{j-s}} \pi_0 & j = c+1, \dots \end{cases}$
L	$\frac{\rho}{1-\rho}$	$\frac{\rho[1-(c+1)\rho^c+c\rho^{c+1}]}{(1-\rho^{c+1})(1-\rho)}$	$L_q + \rho$
L_q	$\frac{\rho^2}{1-\rho}$	$L - L_s$	$\frac{P\{j \geq s\}\rho}{1-\rho}$
L_s	ρ	$1 - \pi_0$	ρ
W	$\frac{1}{\mu-\lambda}$	$\frac{L}{\lambda(1-\pi_c)}$	L/λ
W_q	$\frac{\lambda}{\mu(\mu-\lambda)}$	$\frac{L_q}{\lambda(1-\pi_c)}$	$\frac{P\{j \geq s\}}{s\mu-\lambda}$
W_s	$1/\mu$	$1/\mu$	$1/\mu$

Also, for an $M/M/1/s/FCFS/\infty$ queuing system

$$P\{W > t\} = e^{-\mu t} \left\{ 1 + P\{j \geq s\} \frac{1 - \exp[-\mu t(s-1-s\rho)]}{s-1-s\rho} \right\}$$

$$P\{W_q > t\} = P\{j \geq s\} \exp[-s\mu t(1-\rho)]$$

is a bounded operator.

- (c) Use the Fourier transform to find a general solution of

10 marks

$$\frac{d^2 y}{dt^2} - \frac{dy}{dt} = f(t),$$

where

$$f(t) = \begin{cases} te^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

Hint: show that a particular solution of the above equation has the following Fourier transform:

$$\hat{y}_p = \tilde{f}_1 + \tilde{f}_2 \tilde{f}_3, \quad \text{where}$$

$$\tilde{f}_1 = \frac{-i}{\theta(1+\theta^2)}, \quad \tilde{f}_2 = \frac{1}{1-i\theta}, \quad \tilde{f}_3 = \frac{1}{1+\theta^2},$$

then use the table of transforms to find f_1 , f_2 , f_3 and thus y .

- (d) By using a table of transforms and/or direct integration find the Laplace transform of the functions $f_1(t)$ and $f_2(t)$ which are zero everywhere except

4 marks

$$f_1(t) = \frac{\sin(t)}{t}, \quad t \geq 0,$$

$$f_2(t) = (t - \pi)e^{-(t-2\pi)}, \quad t \geq \pi.$$

What is the Laplace transform of the function $f(t) = f_1(t) + f_2(t)$?

Question 4.

- (a) Answer the following questions in one or two sentences each.

7 marks

- i. Explain what is meant by the term “time series” and give an example.
 - ii. Write down the general form of an $ARIMA(0, 1, 1)(1, 0, 0)^{12}$ model using backshift operators.
 - iii. Explain why it is unlikely that an $AR(10)$ model would ever need to be fitted to a set of data.
 - iv. Explain the *purpose* of taking differences of a dataset in time series analysis.
- (b) A particular time-series $\{X_t\}$ is to be modelled. First differences are taken to produce the series $\{Y_t\}$. The series $\{Y_t\}$ is then modelled using the equation

$$Y_t = 0.3Y_{t-1} + 0.6\epsilon_{t-12} + \epsilon_t,$$

where $\epsilon_t \sim N(0, \sigma^2)$.

- i. Develop a formula for forecasting one-step ahead using this model, and write a short paragraph explaining how to use the formula.
 - ii. Sketch a possible sample autocorrelation function and sample partial autocorrelation function that may have resulted in the selection of the above model for the series $\{Y_t\}$. Briefly explain the features of your sketches that are relevant. (The sketches are to show the important *features* of the possible ACF and PACF, and not necessarily accurate numerical quantities.)
- (c) In his book *Statistics in Biology*, C. I. Bliss gives a data set which discusses the activity of individual fiddler crabs under constant conditions for 225 consecutive hours. The crabs are observed, and the mean amount of activity per hour is measured (in minutes per hour). The data set analysed is $\{W_t\}$, the logarithm (to base 10) of the mean minutes per hour plus one. The data series $\{W_t\}$ is plotted in Figure 2. To analyse this data, the sample ACF and sample PACF were constructed, and are shown in Figure 3.

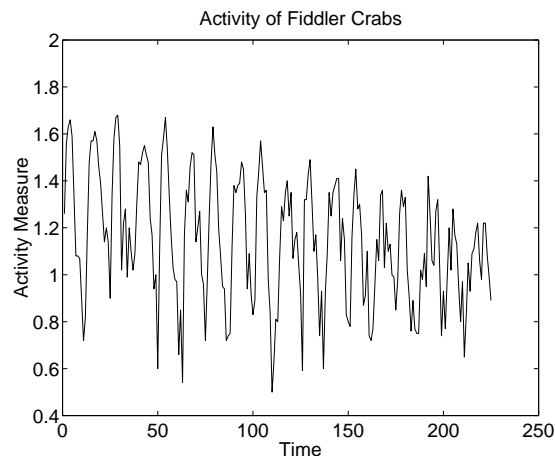


Figure 2: The Fiddler Crab Data Series, $\{W_t\}$

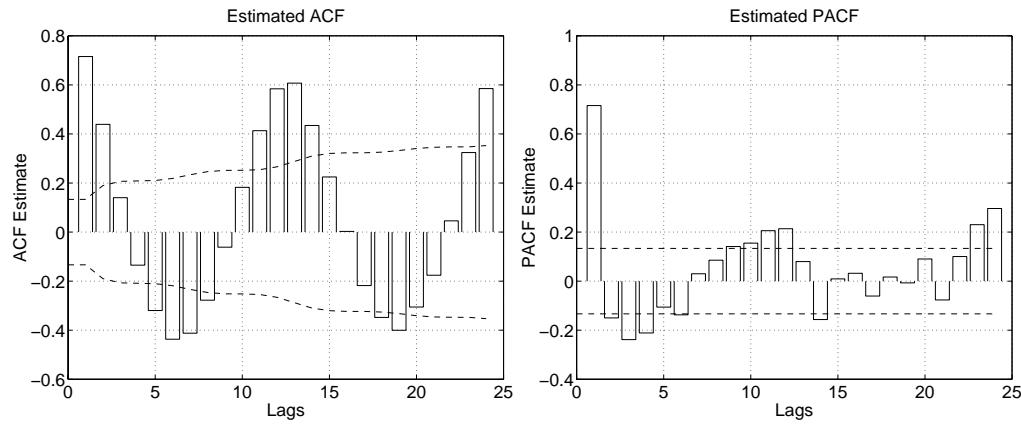


Figure 3: For the Fiddler Crab Data, (a) the ACF and (b) the PACF

- i. Suggest a possible model which could be used to fit this series, justifying your answer.
- ii. List as many ways as possible to check the adequacy of your fitted model, explaining briefly how they assist.

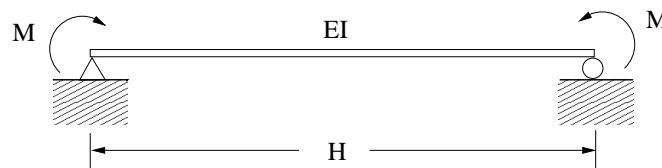
Question 5.

- (a) The differential equation describing the sideways deflection of a beam supported at either end (see the following figure) is 15 marks

$$EI \frac{d^2 y}{dx^2} - M(x) = 0,$$

where the coefficient E is the constant elastic modulus, I the constant area moment, and $M(x)$ is the bending moment.

For the questions below assume that the overall length of the bar is 1 metre, $E = 10^6 \text{ N/cm}^2$, $I = 10^3 \text{ cm}^4$, and the bending moment $M(x) = M_0$, a constant, with $M_0 = 10^6 \text{ N.cm}^2$.



- i. Divide the beam into 2 equal elements. Construct the two equations that describe linear variation in the sideways deflection of the bar (the bar has no deflection at either end).
- ii. By minimizing the integral

$$\int_0^H \frac{EI}{2} \left(\frac{dy}{dx} \right)^2 - My \, dx$$

calculate the deflection of the bar in the middle.

- iii. Now use the weighted residual method to solve for the deflection of the bar in the middle.
- (b) Consider a multigrid discretisation of the equation for the steady state temperature distribution in a long, thin rod, 10 marks

$$-Au''(x) + Bu(x) = 0, \quad 0 < x < 1, \quad A, B > 0;$$

with the rod heated at one end the boundary conditions become $u(0) = T_0$ and $u(1) = T_1$, $T_0 > T_1$. Using this equation and, where necessary, diagrams describe briefly the following:

- i. the residual equation;
- ii. the restriction operation from a fine grid to a coarse grid;
- iii. the interpolation from a coarse grid to a fine grid;
- iv. the steps necessary for one complete multi-grid V-cycle.