Tutorial 10 | Week of 26 January, 2009

Equivalence Relations and Classes

1. A relationship function F is defined on two real numbers x and y by the condition that:
$$x\ F\ y \quad \leftarrow \quad (float(x) = float(y))$$
where function $float(e)$ returns the floating point representation of e in a particular computer representation.

   Show that F is (the relationship function for) an equivalence relationship.

   Note, however, that in a 16 bit floating point system with a 6 bit characteristic (for example) the computer representations for
$$a = 0.0048523,\ b = 0.0048524,\ c = 0.0048525,\ d = 0.0048675$$
are all the same, namely 0011000100111110. In a computer with this representation the products $a \star d$ and $b \star c$ must necessarily work out the same; but the fact is that the computer representations of the exact products are different: 0010000110001100 and 0010000110001011 respectively. Computer arithmetic must cope with these sorts of problems.

2. **Remainders**
Show that the relation $\sim$ defined on integers by
$$x \sim y \iff x - y \text{ is divisible by } 7$$
is an equivalence relation. Describe what is common to the elements in a particular equivalence class and write an algorithm for the relationship function for $\sim$.

3. **The Integers**
Show that the relation $\sim$ defined, on the Cartesian product $N \times N$, where N is the set of natural numbers, by
$$(a, b) \sim (c, d) \iff a + d = b + c$$
is an equivalence relation. Draw a diagram to illustrate the equivalence classes.

   (Notice that the algorithm for the relationship function is expressed in the equality as in Part 1.)

4. **The Rationals**
Show that the relation $\sim$ defined on the Cartesian product $J \times J$, where J is the set of integers, by
$$(a, b) \sim (c, d) \iff a \times d = b \times c$$
is an equivalence relation. Draw a diagram to illustrate the classes of equivalent fractions.

   (Notice that the algorithm for the relationship function is expressed in the equality as in Part 1.)

5. The real solutions of the equation $f(x) = 0$ are those real numbers, ξ for which $f(\xi) = 0$ is true. We shall say that
$$f \sim g \text{ if, and only if, } f(x) = 0 \text{ and } g(x) = 0 \text{ have the same set of solutions.}$$
Check that $\sim$ is an equivalence relation.

(i) If the functions are linear functions, that is, for example: $f(x) = ax + b$ for some real numbers a and b, draw a graph showing some of the equivalence classes. Describe the equivalence classes in words.

(ii) If the functions are quadratic polynomials, that is, for example: $g(x) = ax^2 + bx + c$ for some real numbers a, b and c, draw a graph showing some of the equivalence classes. Describe the equivalence classes in words.

(iii) If the functions are polynomials of any degree, draw a graph showing some of the equivalence classes. Describe the equivalence classes in words.