Logic Quantifiers.

Suppose that x represents any element of the set of boys and y an element of the set of girls, and that P is the propositional function for the two place predicate defined by

$P(a,\ b) = 1$ if, and only if, a likes b.

Interpret in words, the expressions:

$\exists x \forall y\ \{P(y,x)\}$: There are some boys that are liked by all the girls.

$\forall y\ \exists x\ \{P(x,y)\}$: Every girl has some boy who likes her.

Use the above symbolism to express the following statements

some girls don't like any boys : $\exists y \forall x\ :\ P(y,x) = 0.$

every boy will like some girl. : $\forall x \exists y\ : P(x,y) = 1$

Loop equivalent proved by Mathematical Induction

1. Consider the algorithm listed below

   [0] $p = Q(n)$
   [0.5] NB. n non-negative integer
   [0.5] NB. p positive integer
   [1] $p \leftarrow 2$
   [2] $k \leftarrow 1$
   [3] **While** $k < n$
   [3.1] $k \leftarrow k + 1$
   [3.2] $p \leftarrow p + 3 \times (k - 1)$

   Trace this algorithm for several small values of n, say $n = 0, 1, 2, 3, 4, 5$ and make a table of output values for various n.

n	$Q(n)$
0	2
1	2
2	5
3	11
4	20
5	32
6	47
7	65
$\vdots$	$\vdots$

   Trace table for $Q(5)$ is shown below. Trace tables for smaller values of n are simply truncations of this table with a different first column and the test $k < n$ failing earlier.

Step	n	p	k	$k < n$	Step	n	p	k	$k < n$
[0]	5	-	-	-	[3.2]	5	11	3	1
[1]	5	2	-	-	[3]	5	11	3	1
[2]	5	2	1	-	[3.1]	5	11	4	1
[3]	5	2	1	1	[3.2]	5	20	4	1
[3.1]	5	2	2	1	[3]	5	20	4	1
[3.2]	5	5	2	1	[3.1]	5	20	5	1
[3]	5	5	2	1	[3.2]	5	32	5	1
[3.1]	5	2	3	1	[3]	5	32	5	0

The function terminates when the entry in the last column is 0 and the output is then the number in the third column.

$P(n)$ is the proposition that the output, $Q(n)$, from the above algorithm is given by $An^2 + Bn + C$ for some A, B, and C.

Find values of A, B, C which will return the results found for the tabulated values.

Since $Q(0) = 2$ or C from the formula, we must have $C = 2$ and so
$$Q(n) = An^2 + Bn + 2.$$
Since $Q(1) = 2$ and, from the formula, $A + B + 2$, we must have $A + B = 0$ or $B = -A$.
Hence: $$Q(n) = A(n^2 - n) + 2.$$
Since $Q(2) = 5$ and $2A + 2$ from the formula, we must have $2A + 2 = 5$ or $A = \frac{3}{2}$, and
$$Q(n) = \tfrac{3}{2}n(n-1) + 2.$$

Prove that proposition $P(n)$ is true for all natural numbers n.

Certainly P_0 is true. So let us assume that the proposition is true for some value m, i.e the output from the loop algorithm is given by $\frac{3}{2}m(m-1) + 2$ and see what the output would be if the input to the algorithm had been $m + 1$.

After m passes through the loop the trace table's last line would be the first line shown below, assuming the result in p is obtained from the formula (if we had been calculating $Q(m)$ the first column would show m and the last would show 0 at this stage).

Step	n	p	k	$k < n$
[3]	$m+1$	$\frac{3}{2}m(m-1) + 2$	m	1
[3.1]	$m+1$	$\frac{3}{2}m(m-1) + 2$	$m+1$	1
[3.2]	$m+1$	$\frac{3}{2}m(m-1) + 2 + 3((m+1) - 1)$	$m+1$	1
[3]	$m+1$	$\frac{3}{2}m(m-1) + 2 + 3m$	$m+1$	0

The final output for $Q(m + 1)$ is then

$$\tfrac{3}{2}m(m-1) + 3m + 2 = \tfrac{3}{2}m[(m-1) + 2] + 2 = \tfrac{3}{2}m(m+1) + 2 = \tfrac{3}{2}[m+1][(m+1) - 1] + 2$$

and, as we see, could have been obtained from the formula if $Q(m)$ was obtainable from the formula. By the Principle of Mathematical Induction then the result can be obtained from the formula for any n.

Find the value of $Q(2005)$.

$$Q(2005) = 2 + 3 * 1004 * 2007 = 6045086$$

2. Consider the algorithm

    [0] $p = T(n)$
    [1] $p \leftarrow 3$
    [2] $k \leftarrow 1$
    [3] **While** $k < n$
      [3.1] $p \leftarrow p + 5$
      [3.2] $k \leftarrow k + 1$

   Propose a formula for the output, $T(n)$, from the above algorithm and prove your conjecture.

   Executing the algorithm, we see that $T(0) = 3$; $T(1) = 3$; $T(2) = 8$; $T(3) = 13$. With every pass through the loop the increment in the output is 5, so the conjecture is that $T(n) = 3 + 5(n - 1)$.

   Clearly the formula holds for $n = 0$ (and 1 and 2).
   Assume that the formula holds for $n = m$ and let us calculate $T(m+1)$ from the algorithm. After $m - 1$ passes through the loop, the value of p will be, by assumption, $3 + 5(m - 1)$ (for $T(m)$). Executing line [3.1] will then give the value of p as $3 + 5(m - 1) + 5$, that is $3 + 5m$ or $3 + 5[(m + 1) - 1]$. So the formula holds for $n = m + 1$ as well and, by induction, $\forall n$.