Tutorial 7 — Week of 5 January, 2009

Logic Quantifiers.

Suppose that x represents any element of the set of boys and y an element of the set of girls, and that P is the propositional function for the two place predicate defined by

$$P(a,\ b) = 1 \text{ if, and only if, } a \text{ likes } b.$$

(i) Interpret in words, the expressions:

$$\exists x \forall y\ \{P(y,x)\}$$

and

$$\forall y\ \exists x\ \{P(x,y)\}.$$

(ii) Use the above symbolism to express the following statements

some girls don't like any boys

and

every boy will like some girl.

Loop equivalent proved by Mathematical Induction

1. Consider the algorithm listed below

   [0] $p = Q(n)$
   [0.5] NB. n non-negative integer
   [0.5] NB. p positive integer
   [1] $p \leftarrow 2$
   [2] $k \leftarrow 1$
   [3] **While** $k < n$
   [3.1] $k \leftarrow k + 1$
   [3.2] $p \leftarrow p + 3 \times (k - 1)$

   Trace this algorithm for several small values of n, say $n = 0, 1, 2, 3, 4, 5$ and make a table of output values for various n:

n	$Q(n)$
0	2
1	...
2	...
⋮	⋮

   Each trace table should look something like this (for $Q(5)$):

Step	n	p	k	$k < n$
[0]	5	-	-	-
[1]	5	2	-	-
[2]	5	2	1	-
[3]	5	2	1	1
[3.1]	5	2	2	1
[3.2]	5	5	2	1
[3]	5	5	2	1
⋮	⋮	⋮	⋮	⋮

The function terminates when the entry in the last column is 0 and the output is then the number in the third column.

Notice that the trace table for $n = 5$ replicates the table for $n = 4$ except that the second column holds 5 rather than 4 and that, the last column entry not being 0 when k is 4, there are three more steps. In general, the table for $Q(m + 1)$ differs from that for $Q(m)$ only in its second column until it comes to the line shown below.

Step	n	p	k	$k < n$
[3]	$m+1$	$Q(m)$	m	1
[3.1]	$m+1$	$Q(m)$	$m+1$	1
[3.2]	$m+1$	$Q(m) + 3((m+1) - 1)$	$m+1$	1
[3]	$m+1$	$Q(m) + 3m$	$m+1$	0

$P(n)$ is the proposition that the output, $Q(n)$, from the above algorithm is
$Q(n) = An^2 + Bn + C$ for some $A,\ B$, and C.
Find the values of $A,\ B$, and C which will return the correct results for small values of n.
Extend your table to show these values

n	$Q(n)$	$An^2 + Bn + C$
0	2	2
1	$\dots$	$\dots$
2	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$

Prove that proposition $P(n)$ is true for all natural numbers n.
Find the value of $Q(2008)$.

2. Consider the algorithm

   [0] $p = T(n)$
   [1] $p \leftarrow 3$
   [2] $k \leftarrow 1$
   [3] **While** $k < n$
     [3.1] $p \leftarrow p + 5$
     [3.2] $k \leftarrow k + 1$

   Propose a formula for the output, $T(n)$, from the above algorithm and prove your conjecture.