Student name:	Barcode
Student number:	

University of Southern Queensland

Faculty of Sciences

MAT1101
Discrete Mathematics for Computing

End Semester Exam

Assessment No. 3: External

This examination carries 60% of the total assessment for this course.

Examiner: W. Spunde; Moderator: H. Butler

Examination date: February — Semester 3, 200?

Time allowed: Perusal: Ten (10) minutes
Working: Two (2) hours

SOLUTIONS

Question 1.

25 marks

(a) The table below shows *decimal* equivalents of the ASCII codes for the printable characters from the first 128 standard symbols. Add the row number to the column number of any particular symbol to get its decimal equivalent. The decimal equivalent for a space is 32. 127 is the decimal equivalent of the ASCII code for null.

$r+c$	0	1	2	3	4	5	6	7	8	9
30				!	"	#	$	%	&	′
40	(	)	*	+	,	-	.	/	0	1
50	2	3	4	5	6	7	8	9	:	;
60	<	=	>	?	@	A	B	C	D	E
70	F	G	H	I	J	K	L	M	N	O
80	P	Q	R	S	T	U	V	W	X	Y
90	Z	[	\	]	∧	_	‘	a	b	c
100	d	e	f	g	h	i	j	k	l	m
110	n	o	p	q	r	s	t	u	v	w
120	x	y	z	{	|	}	∼			

4 marks (i) Find the character string represented by the (8-bit) ASCII code

`00101110011000110110111101101101`

The individual characters come from each group of 8 bits:

`0010 1110   0110 0011   0110 1111   0110 1101`

which are represented in hex as 2E, 63, 6F, 6C. Converting to decimals, since the table is set out in decimal equivalents, we have 46, 99, 111, 109. From the table, we get the characters:

.com

4 marks (ii) Write down the (8-bit) ASCII code for `M01`

The code for M has decimal equivalent 77 from the table. In binary 77 = 64 + 8 + 4 +1 is 1001101, so the 8-bit ASCII code is 01001101. Numerals are preceded by the code 0011 followed by their binary representation — or, from the table 0 has decimal equivalent 48 = 30_{16} = 0011 0000 and 1 has decimal equivalent 49 = 32 + 16 + 1 = 110001, or in 8-bits 0011 0001. Thus, M01 has ASCII code

010011010011000000110001

(b) If every integer must be stored in 16 bits in a computer with the same number of negative and non-negative integers, find:

3 marks (i) the range of integers that can be represented;

If there are 16 bits the the largest number is $2^{16}-1$ since 2^{16} requires 17 bits. From 0 to $2^{15}-1$ will represent positive integers, but from 2^{15} to $2^{16}-1$ will represent the integers from -2^{15} to -1. So the range of integers that can be represented is

$$(-2^{15}=) \quad -32768 \leq n \leq 32767 \quad (=2^{15}-1)$$

3 marks (ii) the integers represented by
1111111111111111, 1111111111111000 and 0001000000100010.

clearly 1111111111111111 is the last representable number and it represents -1.

1111111111111000 = 1111111111111111 − 111 = −1 − 7 = −8

0001000000100010 = 4096 + 32 + 2 = 4130

(c) If floating point numbers are represented in 16 bits, of which 7 are assigned to the characteristic and the bias for a k-bit characteristic is $2^{k-1}-1$, find:

4 marks (i) the computer representation of -5.45.

Since the mantissa will have 8 bits, an 8-bit binary representation of 5.45 is required. Now,

4 + 1 + 0.25 + 0.125 + 0.0625 + 0.03125 = 5.46875

so the 8-bit binary representation for 5.45 is 101.01110 and the normalized form is 0.10101110×2^3.

Since the bias is 2^6-1 or, in binary, 1000000 - 1 and the exponent (in binary) is 11, the characteristic (exponent + bias) is 1000011-1 = 1000010. The sign bit is 1 and the floating point representation is

1100001010101110

3 marks (ii) the smallest positive real number that can be represented in this system.

The smallest positive real number is 0000000010000000 and since the characteristic is 0, the exponent is the negative of the bias, i.e.$-(2^6-1)=-63$. The normalized form of the number is therefore $0.1\times 2^{-63}=2^{-64}$ or in decimal notation:

5.42101×10^{-20} **= 0.0000000000000000000542101**

4 marks (iii) the range of numbers represented by 1100100111010110.

The characteristic is 1001001 and the bias is 1000000 - 1 and consequently the exponent is 0001001+1 = 0001010 or in decimal notation 10. The normalized binary form of the absolute value of number is therefore 0.11010110×2^{10} that is, 1101011000 or 512 + 256 + 64 + 16 + 8 = 856. The sign bit is -1 so the number is negative, -856.

The same characteristic and mantissa will represent all numbers with absolute value between 856 and the number with normalized form 0.11010111×2^{10}, ***i.e.*** **1101011100 or 860.**

Thus the range of numbers represented is

$$-860 < x \leq -856$$

Question 2.

25 marks

5 marks (a) Describe, with examples, what is meant by an *in-fix* syntax for a a function. What other forms of function syntax can be found in the mathematical literature? Give at least one example of each form.

In-fix syntax is a function call in which inputs appear to the left and right of the function name. The arithmetic operators all have this syntax, for example, $x + y$, $x \times y$, $x \ mod \ y$. **The classical power notation, as for example in** x^2 **and** x^y, **is often written on a computer command line as an in-fix function,** ***e.g.*** $x \wedge 2 \quad x \wedge y$.

The most common function syntax in mathematics is pre-fix notation as in $f(x)$ **or just** fx **; for example** $\cos(x^2)$, $\sin x$, $\log x$. **Post-fix notation, as in** $x!$, **is less common and some notations where the data is inside the function name, as in** $|x|$, $[x]$, $a : b : c$ **are also seen.**

(b) In a particular spreadsheet the syntax for the `ceiling` function is

$$\texttt{N} = \texttt{CEILING}(\texttt{n}, \texttt{m})$$

where `n` is any real number and `N` is the smallest multiple of `m` larger than `n`.

1 mark (i) If `h`← 23.47, `k`← 7 and `z`← `CEILING(h,k)`, what number is stored in the variable `z`?

The multiples of 7 are 7, 14, 21, 28 , . . . and the smallest of these that is larger than 23.47 is 28.

2 marks (ii) What numbers are returned by

$$0.1 \star \texttt{CEILING}(23.47 - 0.5, 1) \quad \text{and} \quad 0.01 \star \texttt{CEILING}(234.7 - 0.5, 1)$$

The multiples of 1 are all the integers. The smallest integer bigger than 22.97 is 23; and the smallest integer bigger than 234.2 is 235. Multiplying these two numbers by 0.1 and 0.01 gives

2.3 2.35

5 marks (iii) Using function CEILING in the algorithm, write pseudo code for a function that, given inputs n and m and the same syntax as CEILING, returns the value of n rounded to m decimal places.

We notice that the results in the previous question were the values of 2.347 rounded to one and two decimal places and so we may deduce that the value of a decimal, d, rounded to n places will be given by

$$\mathbf{10}^{-n} \times \texttt{CEILING}(10^n \times (d - 0.5), 1)$$

Pseudo-code for a function with syntax similar to CEILING would then be:

```
[0] R = ROUND(n, m)
  [0.1]NB. n any decimal; m a natural number
  [0.2]NB. R is n rounded to m decimal places
[1] R ← 10^-m × CEILING(n × 10^m − 0.5, 1)
  [1.1]NB. or :
  [1.2]NB.   CEILING(n − 0.5 × 10^-m, 10^-m)
```

2 marks (iv) Using your function from part (iii), write the instruction that will round 2.765001 to 2 decimal places. What number will it return?

ROUND(2.765001, 2) = 2.77

(c) Starting with any positive fraction, f less than 1, the next number in a sequence is obtained by multiplying the difference between the last number in the list and the square of that number by 1.5. For example the next number after 0.4 will be $0.36 = 1.5 \star (0.4 - 0.16)$.

Function GROWTH takes inputs f and n and returns the n^{th} number in the sequence (counting from 0).

8 marks (i) Illustrate the difference between an iteratively defined function and a recursively defined function by writing pseudo code for function GROWTH with *two* different algorithms, one iterative and one recursive.

We have used a for-loop to implement the iterative solution.

```
[0] R = GROWTH(f, n)
  [0.1]NB. f any decimal; n a natural number
  [0.2]NB. R is the n^th term in a computed sequence
  [0.3]NB. using an iterative algortihm
[1] R ← f
[2] for k = 1 to n
  [2.1] R ← 1.5 × (R − R^2)
```

For the recursive solution, we have:

```
[0] R = GROWTH(f,n)
  [0.1]NB. f any decimal; n a natural number
  [0.2]NB. R is the nth term in a computed sequence
  [0.3]NB. using a recursive algortihm
[1] if n = 0
  [1.1] R ← f
  else
  [1.2] a ← GROWTH(f,n − 1)
  [1.3] R ← 1.5 × (a − a²)
```

2 marks (ii) What are the essential features of any recursively defined function?

A recursive function will call itself in the process of executing its algorithm and to avoid an infinite loop it must have an executable option which is eventually reached after a certain number of recursive calls.

Question 3.

25 marks

3 marks (a) Determine which of the following statements is a proposition and explain your reasons.

"To be or not to be"

– **no, that is a question.**

"I think therefore I am"

– **dubious — can we test the truth of the assertion?**

"I studied hard for this exam"

– **yes, we can determine whether this is true or not.**

(b) The statement, p :

"You cannot have satisfaction without commitment"

is a compound proposition with component propositions q and r where

$q \equiv$ *"You have satisfaction"*

$r \equiv$ *"You have commitment"*

3 marks (i) Write p in terms of q, r and all or some of the operators $\wedge, \vee$ and $\neg$.

$\neg(q \wedge \neg r)$ — **not (satisfaction and not commitment)**

3 marks (ii) Hence deduce that p can also be written as an implication.

$$\neg(q \wedge \neg r) \equiv \neg q \vee r \equiv q \rightarrow r$$

2 marks (iii) Express the contrapositive of this implication verbally.

The contrapositive is $\neg r \rightarrow \neg q$ **or, in words,**
if you don't have commitment, you won't have satisfaction

4 marks (iv) Write the converse and inverse implications in terms of the component propositions and express each in words.

The converse is $r \rightarrow q$, **or, in words,**
if you have commitment, then you have satisfaction

The inverse is $\neg q \rightarrow \neg r$, **or, in words,**
if you have no satisfaction, then you have no commitment

(c) Suppose that b represents any element of the set, $\mathcal{B}$, of all customers on a database and that g is any element of the set, $\mathcal{G}$, of all goods stocked by a supplier. P is a relationship between sets $\mathcal{B}$ and $\mathcal{G}$ defined by

$b\ P\ g$ if, and only if, customer b has purchased item g.

Interpret the symbolic statements below at left and, then complete the sentences to the right so that they have the same meaning. (Write your completed statements into your examination booklet.)

2 marks (i) $\exists\, b\, \exists\, g\ :\ b\ P\ g$

There is at least one buyer and one good that was purchased by that buyer, so:
At least some of the goods have been purchased by some customers.

2 marks (ii) $\forall\, b\, \exists\, g\ :\ \neg(b\ P\ g)$

Every customer has some goods that he does not purchase, so:
Some of the goods are not purchased by some customers, and in fact, no customer buys every stocked item.

2 marks (iii) $\forall\, g\, \exists\, b\ :\ \neg(b\ P\ g)$

For any item, there is at least one buyer who is not going to purchase it, so:
There are customers who don't purchase every item, in fact, none of the items is purchased by everyone.

[Note that you may have to modify the sentence to get the required meaning. The last two sentences start off in a way that cannot convey the correct meaning. Consequently, to meet the requirements, we begin with a statement that is close to the correct meaning and then modify it to get the precise meaning.]

4 marks (d) Prove that

$$\forall\, n \geq 0: \quad 14^n - 27 \quad \text{is divisible by 13.}$$

For $n = 0$ we have $14^n - 27 = -26$ which is divisible by 13.

Now if for any value of k, $14^k - 27$ is divisible by 13, then for some integer m we have:

$$\begin{aligned} 14^k - 27 &= 13m \\ \text{therefore} & \\ 14(14^k - 27) &= 14 \times 13m \\ 14^{k+1} - 14 \times 27 &= 14 \times 13m \\ 14^{k+1} - 27 - 13 \times 27 &= 14 \times 13m \\ 14^{k+1} - 27 &= 14 \times 13m + 13 \times 27 \\ &= 13 \times (14m + 27) \end{aligned}$$

which shows that $14^{k+1} - 27$ is divisible by 13.

Thus if $14^n - 27$ is divisible by 13 for any value of n then it is also divisible by 13 for the next value of n. Since $14^n - 27$ is divisible by 13 for $n = 0$ we must therefore conclude that it is divisible by 13 for all values of n.

Question 4.

25 marks

1 mark (a) What can be said about a connected graph with n vertices and $n - 1$ edges, where n is a positive integer?

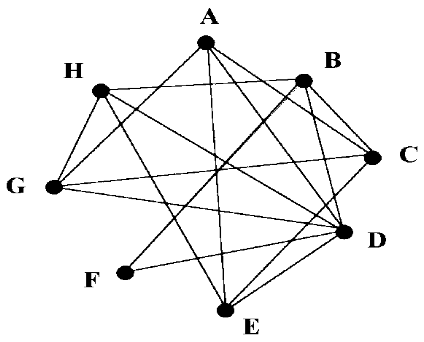
The graph must be a tree.

2 marks (b) What happens to the spanning tree of a graph if an edge (belonging to the graph) is removed from the tree? Explain your answer.

If an edge in a spanning tree is removed, the graph becomes disconnected, because there is only one path between the vertices at the ends of that edge. Therefore there is no way of getting from one of these vertices to the other when that edge is removed.

(c) With reference to the graph below:

3 marks (i) Write down the adjacency matrix for this graph. Based on the analysis of your adjacency matrix, is this graph Eulerian? Explain your answer using the appropriate theorem.



$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

The column (or row) sums in the adjacency matrix are

4, 4, 4, 6, 4, 2, 4, 4

and all vertices therefore have even degree. By Euler's theorem the graph is Eulerian.

2 marks (ii) If the graph is Eulerian, find an Eulerian circuit starting at vertex **B**. If not, adjust the graph by adding a suitable number of edges to make it Eulerian.

Let us begin by making a circuit starting from B:
B–C–A–G–H–D–E–H–B
There are still paths from B and we follow one:
B–D–A–E–C–G–D–F–B
Thus we have one possible Eulerian path covering all 16 edges without repetitions (there are many others):

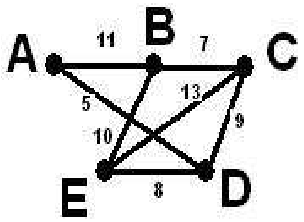
B–C–A–G–H–D–E–H–B–D–A–E–C–G–D–F–B.

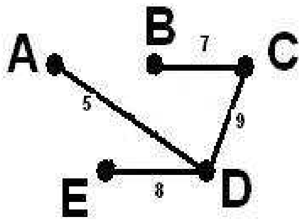
As a partial check, note that since the sum of the degrees is 32, there must be 16 edges in the path each of which must be mentioned in the Eulerian path. Furthermore, if a vertex has degree 6, then it must appear 3 times in the Eulerian path, if degree 4, then twice and if degree 2, then only once. The first and last appearances of B are equivalent.

4 marks (d) Draw a weighted graph represented by the weight matrix

$$W = \begin{bmatrix} 0 & 11 & \infty & 5 & \infty \\ 11 & 0 & 7 & \infty & 10 \\ \infty & 7 & 0 & 9 & 13 \\ 5 & \infty & 9 & 0 & 8 \\ \infty & 10 & 13 & 8 & 0 \end{bmatrix}$$

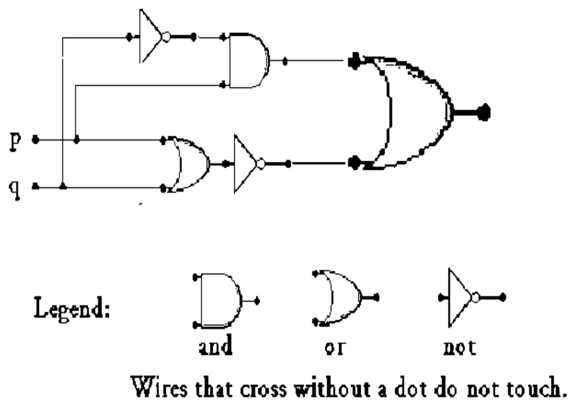
and compute and sketch the minimal spanning tree.





Kruskal's algorithm calls for simply keeping the lowest cost edges unless it would create a cycle and the graph at right shows the result. Minimal spanning tree has cost 29.

(e) A diagram of a logic circuit is shown below.

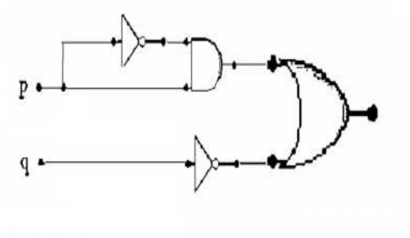


2 marks (i) Write a symbolic expression for the output of the circuit.

$$(\neg q \wedge p) \vee \neg(q \vee p)$$

3 marks (ii) Simplify the expression for the output and draw an equivalent circuit that produces the same output for inputs p and q.

$$\begin{aligned} (\neg q \wedge p) \vee \neg(q \vee p) &\equiv (\neg q \wedge p) \vee (\neg q \wedge \neg p) \\ &\equiv \neg q \wedge (p \vee \neg p) \\ &\equiv \neg q \wedge t \\ &\equiv \neg q \end{aligned}$$



(f) The truth table for a compound proposition with three component propositions a, b, c is shown below.

a	0	0	0	0	1	1	1	1
b	0	0	1	1	0	0	1	1
c	0	1	0	1	0	1	0	1
$g(a, b, c)$	0	0	0	1	0	1	0	1

3 marks (i) Write down the disjunctive normal form for the proposition with truth values given by g.

$$\boldsymbol{g(a, b, c) = a'bc + ab'c + abc}$$

5 marks (ii) Draw the *simplest* logic circuit that produces output for inputs (a, b, c) identical to that of the propositional function g.

Either using an appropriate Karnaugh map or

$$\begin{aligned} \boldsymbol{g(a,b,c)} &= \boldsymbol{a'bc + ab'c + abc} \\ &= \boldsymbol{a'bc + abc + ab'c + abc} \\ &= \boldsymbol{(a' + a)bc + a(b + b')c} \\ &= \boldsymbol{bc + ac} \\ &= \boldsymbol{(b + a)c} \end{aligned}$$