MAT1101 Assignment 2

Solutions

Question 1. [15 marks]

(a) In words, the proposition represented here symbolically by r is:
Even if you fail one of the assignments, you can still pass the course
can be written as the conjunction of two propositions, p and q.

(i) Write verbal expressions for each of p and q (that is, write the component propositions in words) [1 mark]

$p \equiv$ "you fail an assignment" $q \equiv$ "you pass the course".

(ii) Write a symbolic expression for the original statement (that is, write r in terms of p and q). [1 mark]

$r \equiv p \wedge q$.

(iii) Write down the negation of the proposition and show that it can be written as an implication. Express this implication in words.

$$\neg r \;\equiv\; \neg(p \wedge q) \;\equiv\; \neg p \vee \neg q \;\equiv\; p \to \neg q.$$

In words: *if you fail an assignment, then you fail the course.*

(Proposition r, of course, asserts that this is not the case.)

[4 marks]

(b) Express the following implication, A, symbolically in terms of its component propositions:
You cannot pass the course unless you sit for the final examination.
and write in words [2 marks]

With $x \equiv$ "you pass the course $y \equiv$ "you sit the final exam"
proposition A is $\neg y \to \neg x$.

(i) the converse, B, of the implication [1 mark]

$B \equiv \neg x \to \neg y$, which in words is
"if you didn't pass the course, then you didn't sit for the final exam."

(ii) the contrapositive, C, of the implication. [1 mark]

$C \equiv x \to y$, which in words is
"if you passed the course, then you must have sat for the final exam."

(iii) the inverse, D, of the implication. [1 mark]

$D \equiv y \to x$, which in words is
"if you sit for the final exam, then you pass the course."

(iv) Which of A, B, C and D are equivalent to each other? [2 marks]
$A \equiv C$ and $B \equiv D$

(c) Simplify the following compound proposition using either the laws of logic, truth tables or a Karnaugh map
$q \wedge ((\neg q \wedge p) \vee (p \wedge q))$ [2 marks]

$$\begin{aligned} q \wedge ((\neg q \wedge p) \vee (p \wedge q)) &\equiv q \wedge ((p \wedge \neg q) \vee (p \wedge q)) \\ &\equiv q \wedge (p \wedge (\neg q \vee q)) \\ &\equiv q \wedge (p \wedge t) \\ &\equiv q \wedge p \end{aligned}$$

or $$q(q'p + pq) = qq'p + qpq = 0p + pqq = pq$$
or

p	q	$\neg q$	$a \equiv \neg q \wedge$ p	$b \equiv p \wedge q$	$a \vee b$	$q \wedge (a \vee b)$
0	0	1	0	0	0	0
0	1	0	0	0	0	0
1	0	1	1	0	1	0
1	1	0	0	1	1	1

The last column matches the column for $p \wedge q$ which is therefore an equivalent expression.

Question 2. [15 marks]

(a) A number of the form $2^n - 1$ is called the n^{th} Mersenne number. When a Mersenne number is a prime number it is called a Mersenne prime. The largest prime know to man (at this point in time) is a Mersenne prime.

(i) What *is* the largest prime known to man? [1 mark]
$2^{\cdots} - 1$
NB. the value for the 45^{th} Mersenne prime is not published at time of writing. The 44th known Mersenne prime has $p = 32,582,657$.

(ii) How many Mersenne primes are there? [1 mark]
44 for sure and maybe by 2009 a 45^{th} will have been confirmed.

(iii) Justify your answers for parts (i) and (ii). [1 mark]
This information was obtained from the homepage (http://mersenne.org/) of the Great Internet Mersenne Prime Search (GIMPS). The 44th known Mersenne prime has over 9.8 million digits and so it is rather difficult to check its validity. We trust that it's true, because it's only worth one mark. [1 mark]

(iv) What is the binary form of the n^{th} Mersenne number? [1 mark]

The binary form is a string of n 1s: $2^n - 1 \equiv 111\ldots11$ (n bits).

(v) Do you believe that every Mersenne number, $2^n - 1$, with even n is divisible by 3? Justify your answer. [5 marks]

Check out the numbers $2^v - 1$ where v is an even number.

v	2	4	6	8	10	12	14	16	18	20	22
$2^v - 1$	3	15	63	255	1023	4095	16383	65535	262143	1048575	4194303
$\frac{(2^v-1)}{3}$	1	5	21	85	341	1365	5461	21845	87381	349525	1398101

The first eleven even numbers all generate a multiple of 3.

To justify the belief, we can prove that they are all divisible by 3 either by mathematical induction or by noting that the binary representation consists of a whole number of 11_2 *pairs* and consequently 3 times the sum of various powers of 2.

By induction: if P_n is the proposition that the n^{th} even number, v_n generates a multiple of 3, then certainly P_1 is true.

Assume that P_k is true then

$$2^{v_k} - 1 = 3 \times m \qquad \text{for some value } m$$

The next even number after v_k is $v_{k+1} = v_k + 2$ and

$$2^{v_k+2} - 1 = 4 \times 2^{v_k} - 1 = 4 \times (1 + 3 \times m) - 1 \qquad \text{from } P_k$$

and so we have

$$2^{v_{k+1}} = 12 \times m - 3 = 3 \times (4 \times m - 1) = 3 \times M$$

That is, P_{k+1} is true. By induction the proposition P_n is true for all n

(vi) Do you believe that every Mersenne number, $2^p - 1$, where p is prime, is a prime? Justify your answer. [2 marks]

Since there are only 44 or 45 known Mersenne primes and the p values are up in the millions there must be hundreds of thousand of Mersenne numbers with prime exponent that are not prime numbers. We can actually prove our claim by producing one such number.

While it is true that for $p = 2, 3, 5$ and 7, $2^p - 1$ is prime, it is not true for $p = 11$ when $2^{11} - 1 = 2047 = 23 \times 89$. So we have produced a counter-example to the assertion that $2^p - 1$ is prime.

(b) Suppose that P represents any element of the set, $\mathcal{P}$, of all airports and that C is any element of the set, $\mathcal{C}$, of all commercial airlines. $\mathcal{R}$ is the relationship function defined on $\mathcal{P} \times \mathcal{C}$ by

$P \,\mathcal{R}\, C \;= 1$

if, and only if, airport P has scheduled planes from airline C to land there.

Interpret the symbolic statements below at left and complete the sentences to the right in non-technical English so that they have the same meaning.

(i) $\forall\, C\, \exists\, P \;:\; P\, \mathcal{R}\, C \;= 0$
No airline lands at all airports.

(ii) $\exists\, P\, \forall\, C \;:\; P\, \mathcal{R}\, C \;= 0$
Commercial airlines are not scheduled to land at all at some airports.

[2 marks]

(c) The following table lists the degrees of the various vertices of an undirected graph.

V1	V2	V3	V4	V5	V6	V7
12	8	8	12	10	12	8

Which of the following paths could be an Eulerian circuit through the graph.

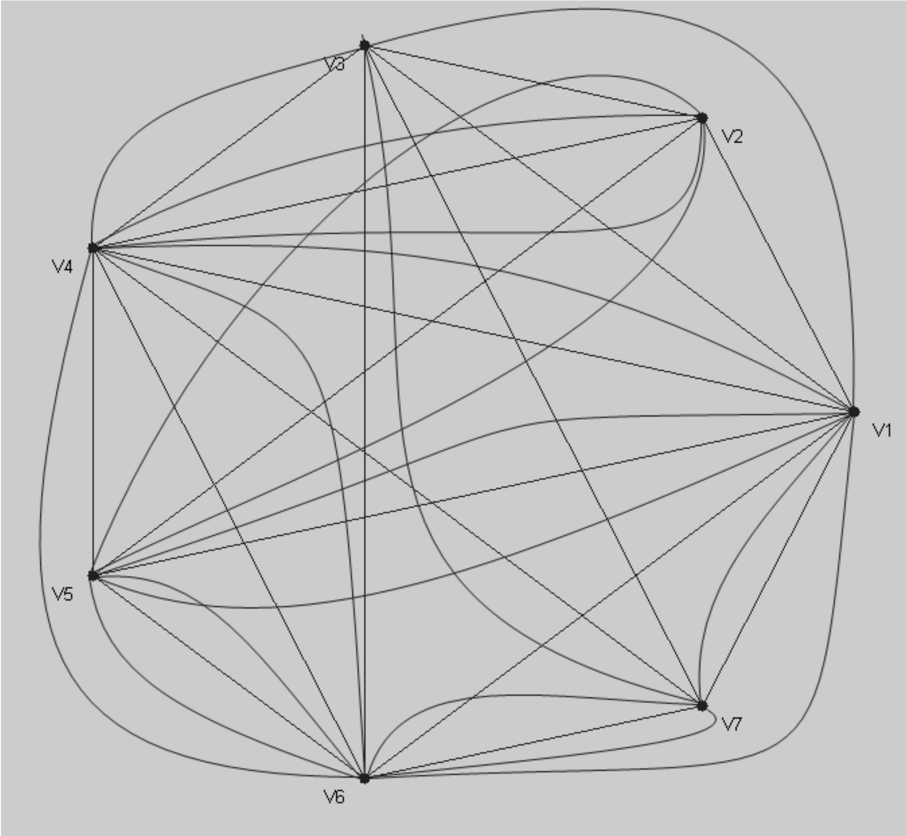
(1) V1-V2-V3-V4-V5-V6-V7-V1
(2) V1-V3-V4-V6-V1-V3-V6-V7-V1-V4-V6-V1-V4-V6-V5-V2-V4-V2-V5-V1-V5-V2-V4-V7-V6-V5-V1-V2-V3-V7-V3-V4-V5-V6-V7-V1
(3) V1-V3-V4-V6-V5-V2-V4-V2-V5-V1-V5-V2-V4-V6-V1-V3-V6-V7-V1-V4-V6-V1-V4-V7-V6-V5-V1-V2-V3-V7-V3-V4-V5-V6-V7-V1
(4) V1-V3-V4-V6-V5-V2-V4-V6-V1-V3-V6-V7-V1-V4-V6-V1-V2-V5-V1-V5-V2-V4-V5-V4-V7-V6-V5-V2-V3-V7-V3-V4-V5-V6-V7-V1
(5) V1-V3-V4-V6-V1-V3-V6-V7-V1-V4-V6-V1-V4-V6-V3-V7-V5-V4-V3-V1-V5-V2-V4-V7-V6-V5-V1-V2-V3-V7-V3-V4-V5-V6-V7-V1

Explain your reasoning. [2 marks]

For an Eulerian path, all edges must be covered once and once only, so from an Eulerian circuit we can calculate the degree of each vertex. For the paths given, these degrees would be:

2, 2, 2, 2, 2, 2, 2
12, 8, 8, 12, 10, 12, 8
12, 8, 8, 12, 10, 12, 8
10, 8, 10, 12, 14, 8, 8
12, 4, 12, 12, 8, 12, 10

Paths (2) and (3) are the only ones with the correct degrees for each vertex. Sketching the Eulerian paths given we get the graph:



and the adjacency matrix:

G	V1	V2	V3	V4	V5	V6	V7
V1	0	1	2	2	3	2	2
V2	1	0	1	3	3	0	0
V3	2	1	0	2	0	1	2
V4	2	3	2	0	1	3	1
V5	3	3	0	1	0	3	0
V6	2	0	1	3	3	0	3
V7	2	0	2	1	0	3	0